Lecture 17
Statistical Models of Images

6.869/6.819 Advances in Computer Vision

Bill Freeman, Phillip Isola

spring 2021

April 21, 2021

The visual system seems to be tuned to a set of images:

Demo inspired from D. Field

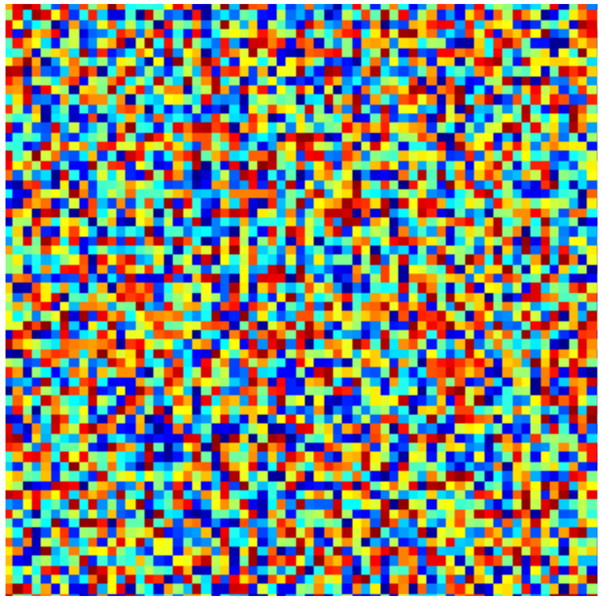
Remember all these images

Was this one of them?

Remember all these images

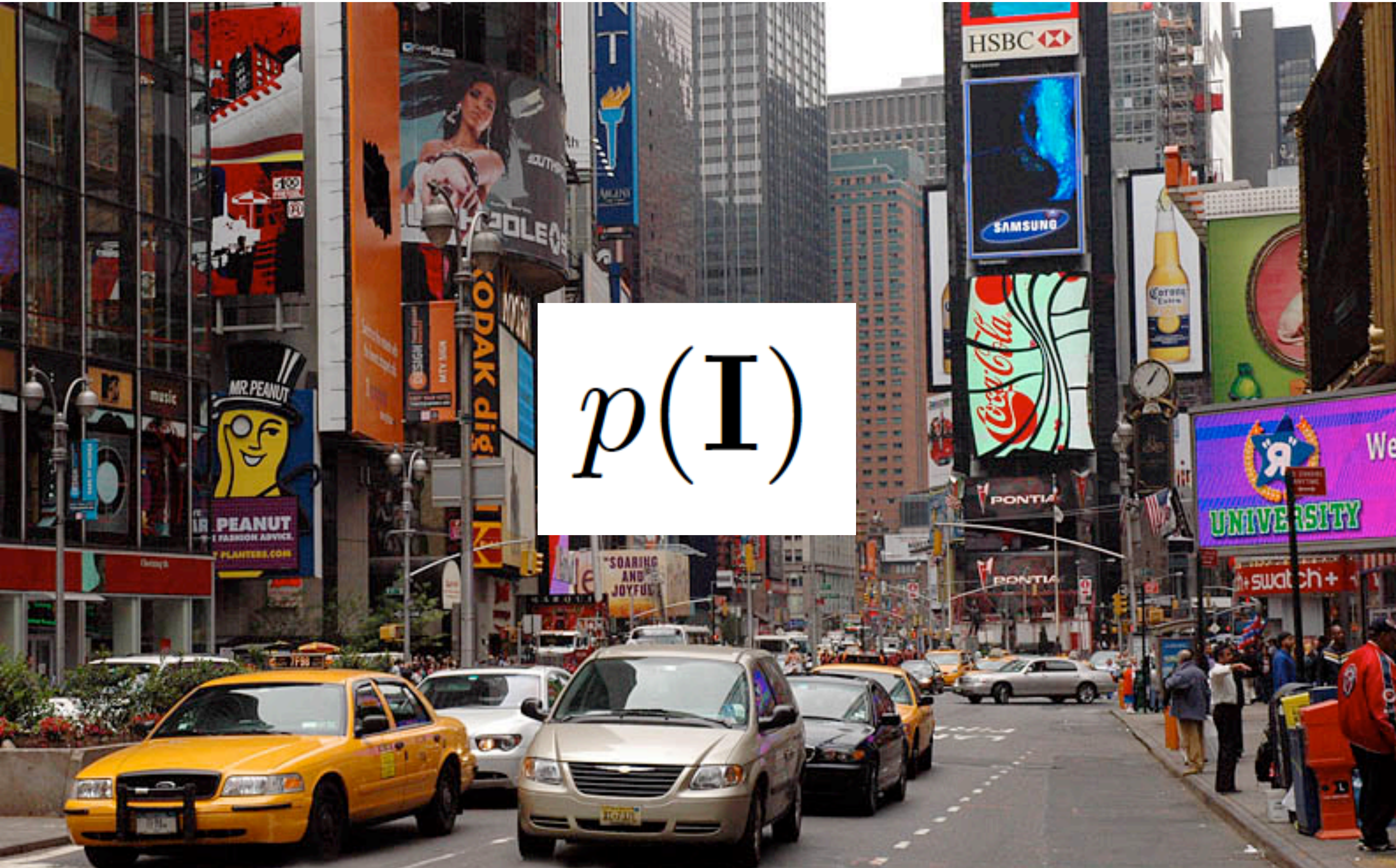
Test 2

Was this one of them?



The visual system is tuned to process structures typically found in the world.

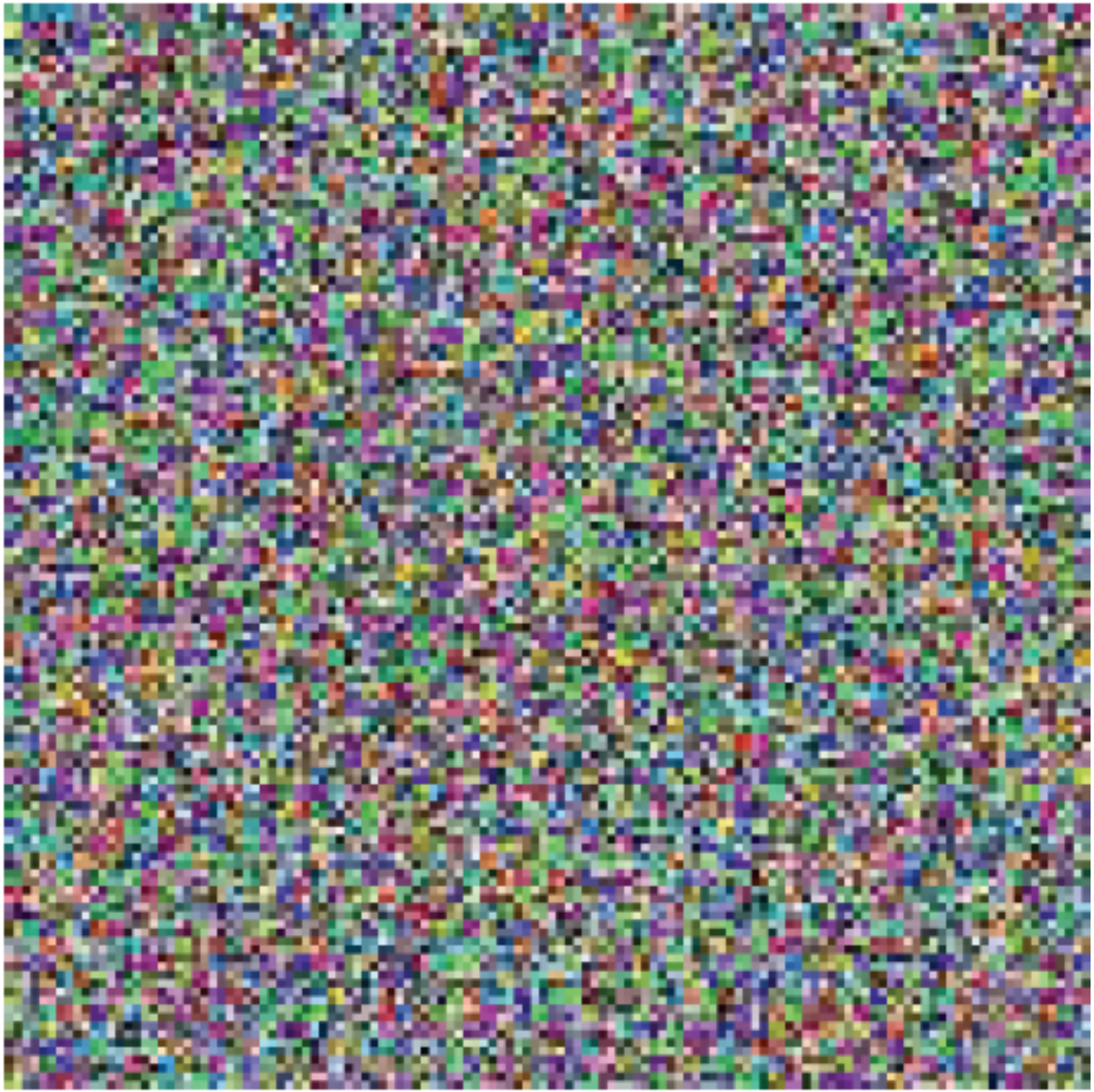
Statistical modeling of images



$$p(\mathbf{I})$$

Visual Worlds

Visual Worlds



Visual Worlds

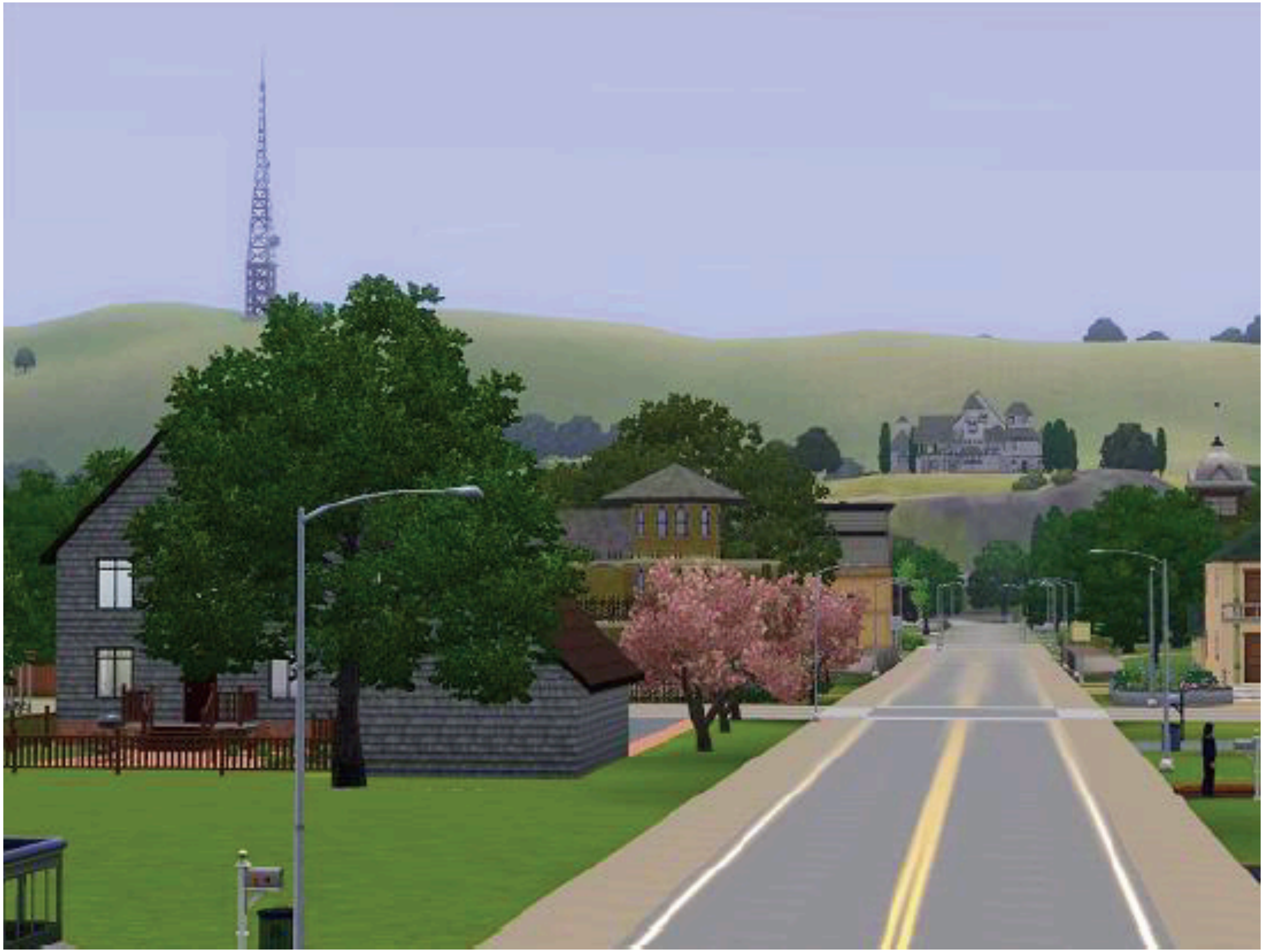
Visual Worlds

Visual Worlds

Visual Worlds

Visual Worlds

Visual Worlds



Visual Worlds

Statistical Image Models

- Gaussian image model
  - image synthesis
  - Wiener filter denoising
- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising
- Non-parametric model
  - image synthesis (Efros and Leung texture model)
  - Non-local means denoising

Statistical Image Models—Readings

Optional additions to the chapter notes.

- Gaussian image model
  - image synthesis
  - Wiener filter denoising
- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising
- Non-parametric model
  - image synthesis (Efros and Leung texture model)
    http://people.eecs.berkeley.edu/~efros/research/NPS/efros-iccv99.pdf
  - Non-local means denoising
    http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.374.7899&rep=rep1&type=pdf

Simoncelli paper
https://pdfs.semanticscholar.org/ee55/814e8705f5e8cf664efb66c31c0ea6372d92.pdf

inspiration for Gatys et al image stylization

Statistical Image Models

- Gaussian image model
  - image synthesis
  - Wiener filter denoising
- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising
- Non-parametric model
  - image synthesis (Efros and Leung texture model)
  - Non-local means denoising

Statistical modeling of images

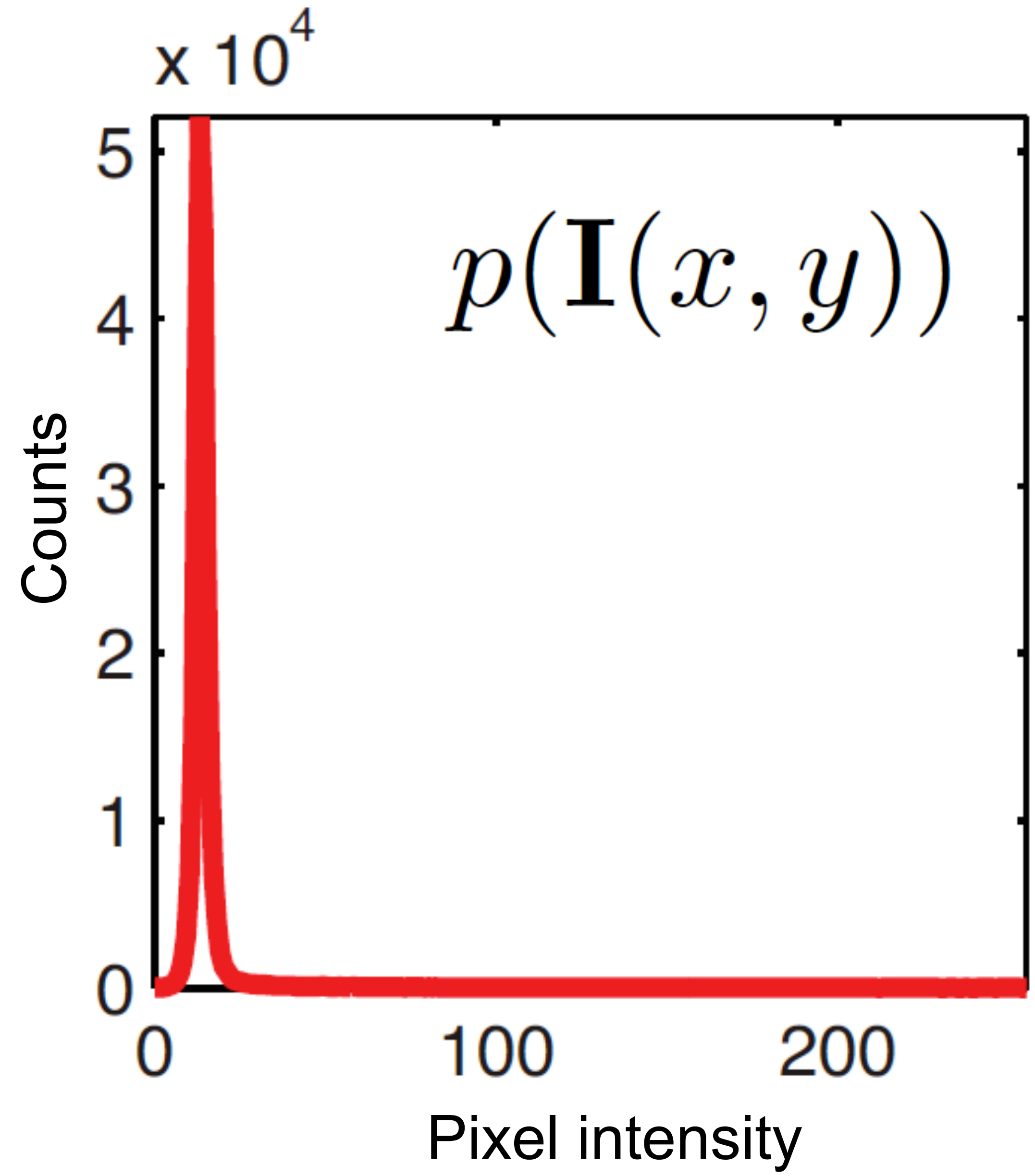
0th image model: independent pixels

$$p(\mathbf{I}) = \prod_{x,y} p(\mathbf{I}(x,y))$$

Assumptions:
- Independence: All pixels are independent.
- Stationarity: The distribution of pixel intensities does not depend on image location.

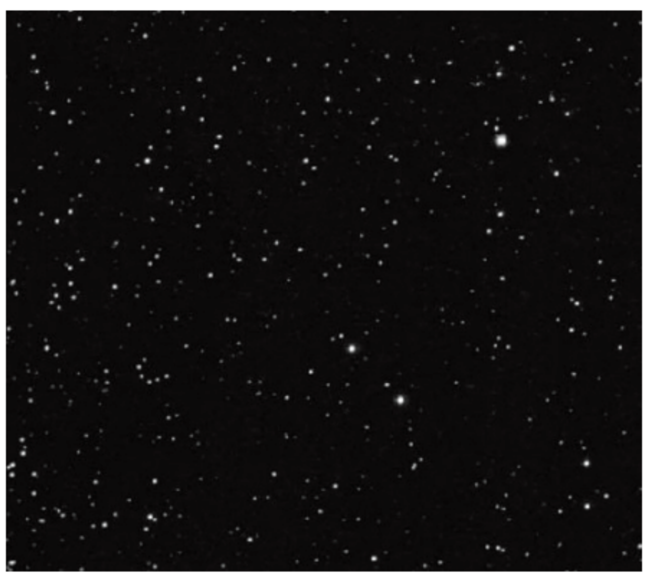
$$p(\mathbf{I}) = \prod_{x,y} p(\mathbf{I}(x,y))$$

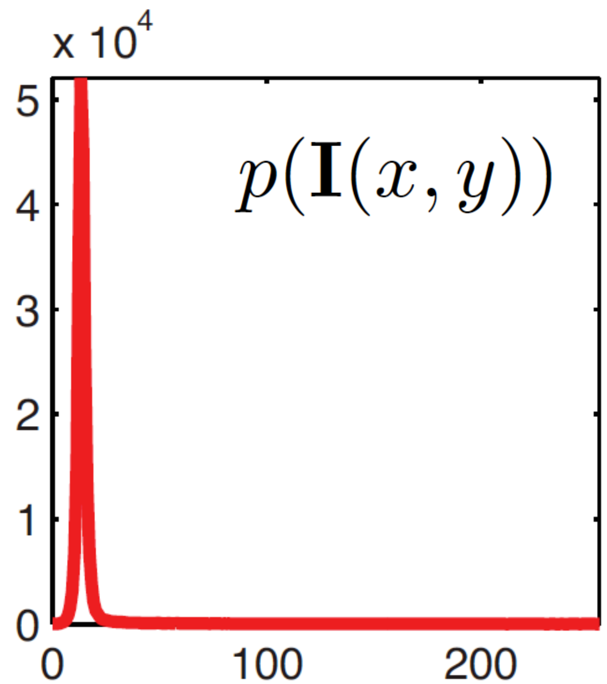
Fitting the model



Sampling new images

$$p(\mathbf{I}) = \prod_{x,y} p(\mathbf{I}(x,y))$$

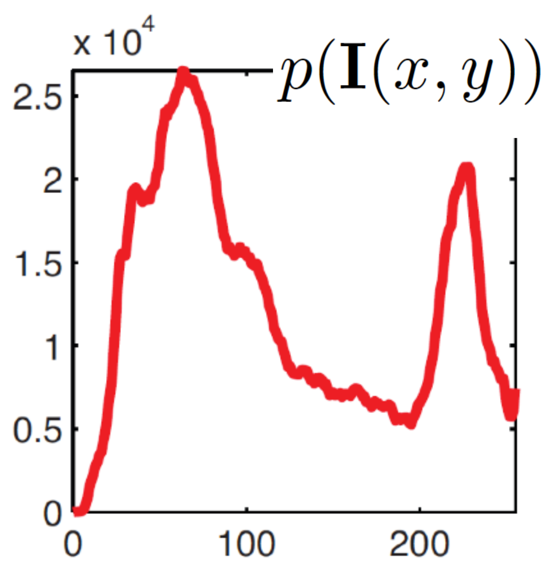


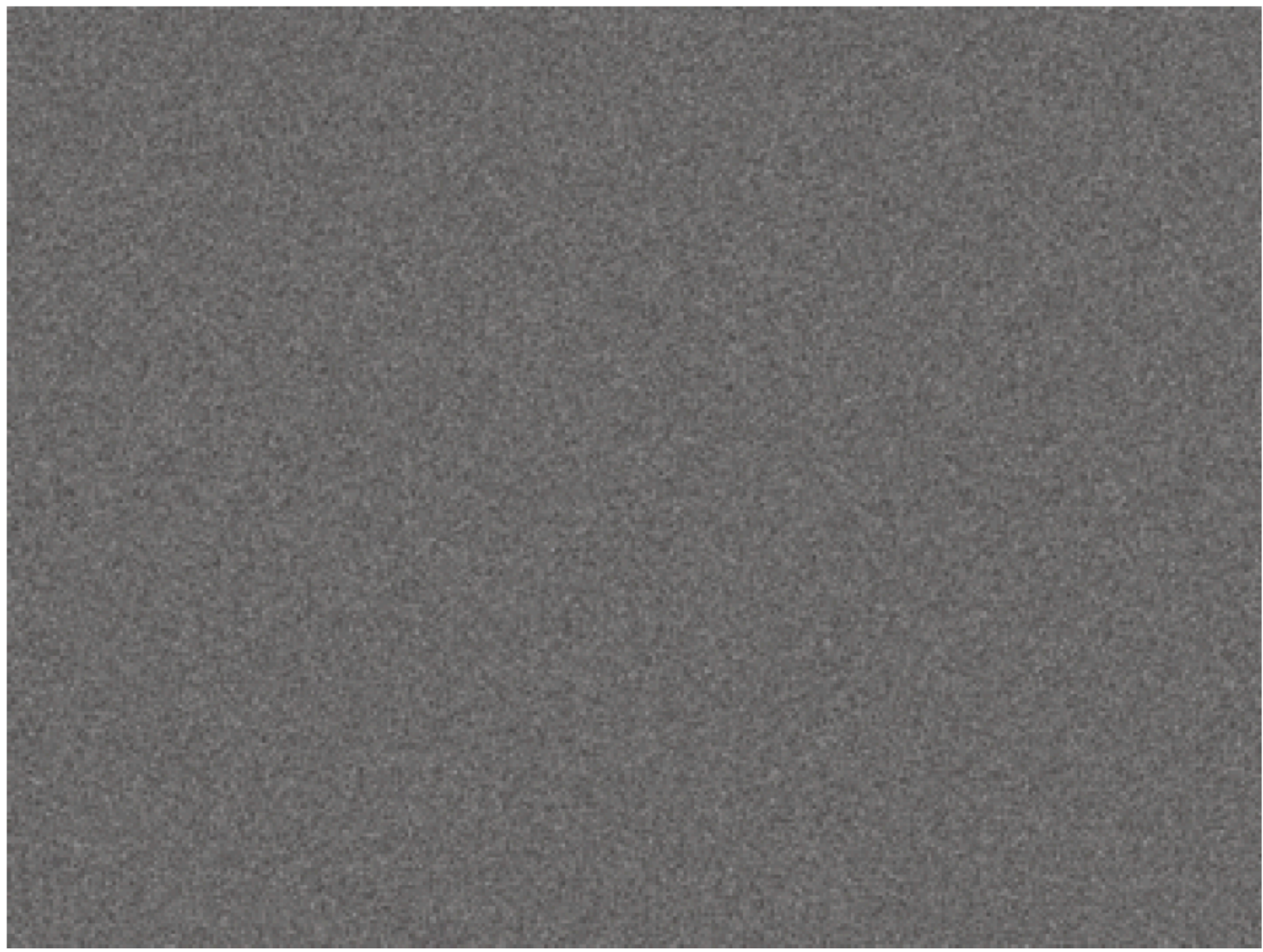


Sample

Sampling new images

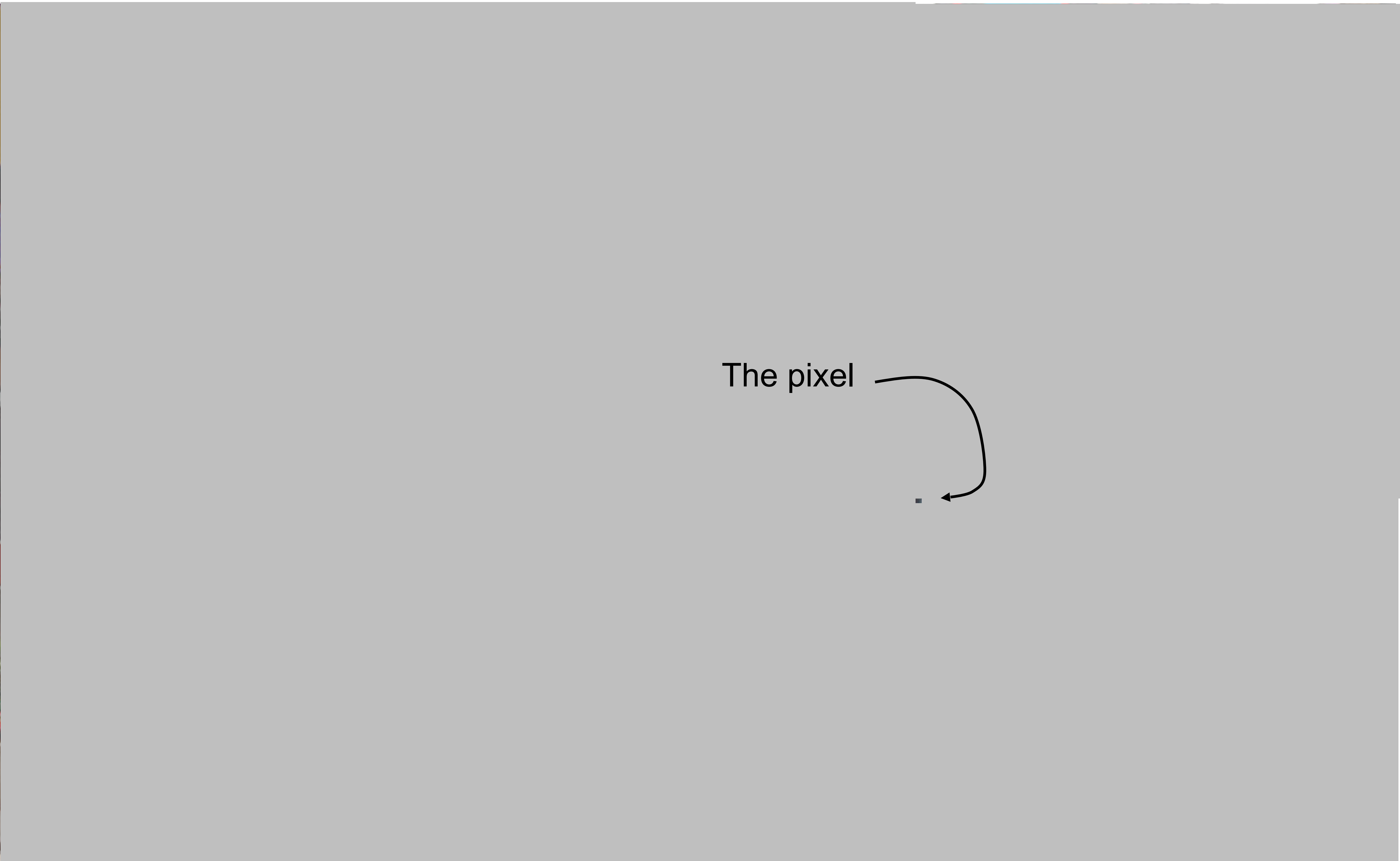
$$p(\mathbf{I}) = \prod_{x,y} p(\mathbf{I}(x,y))$$



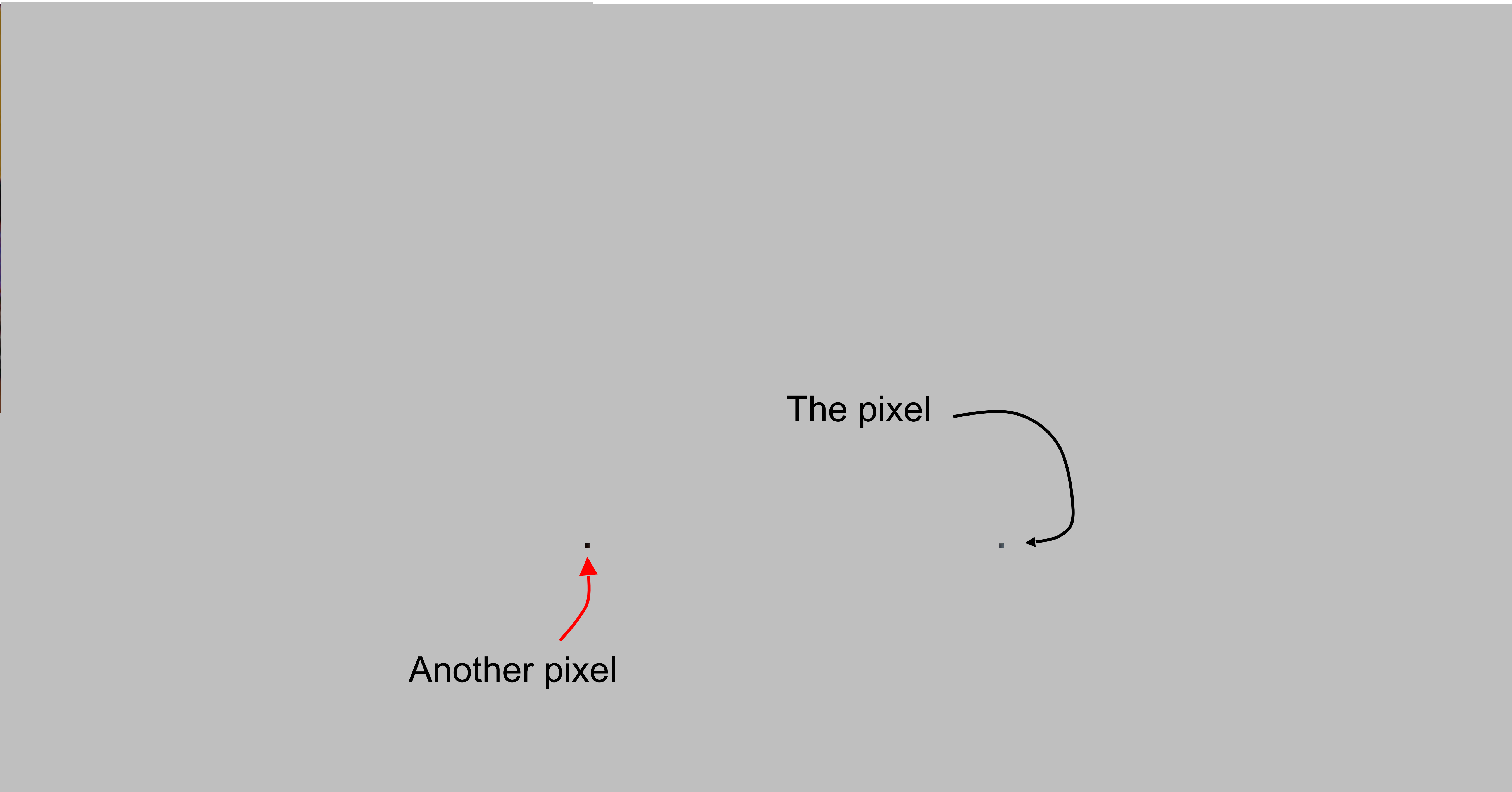


Sample

0th model

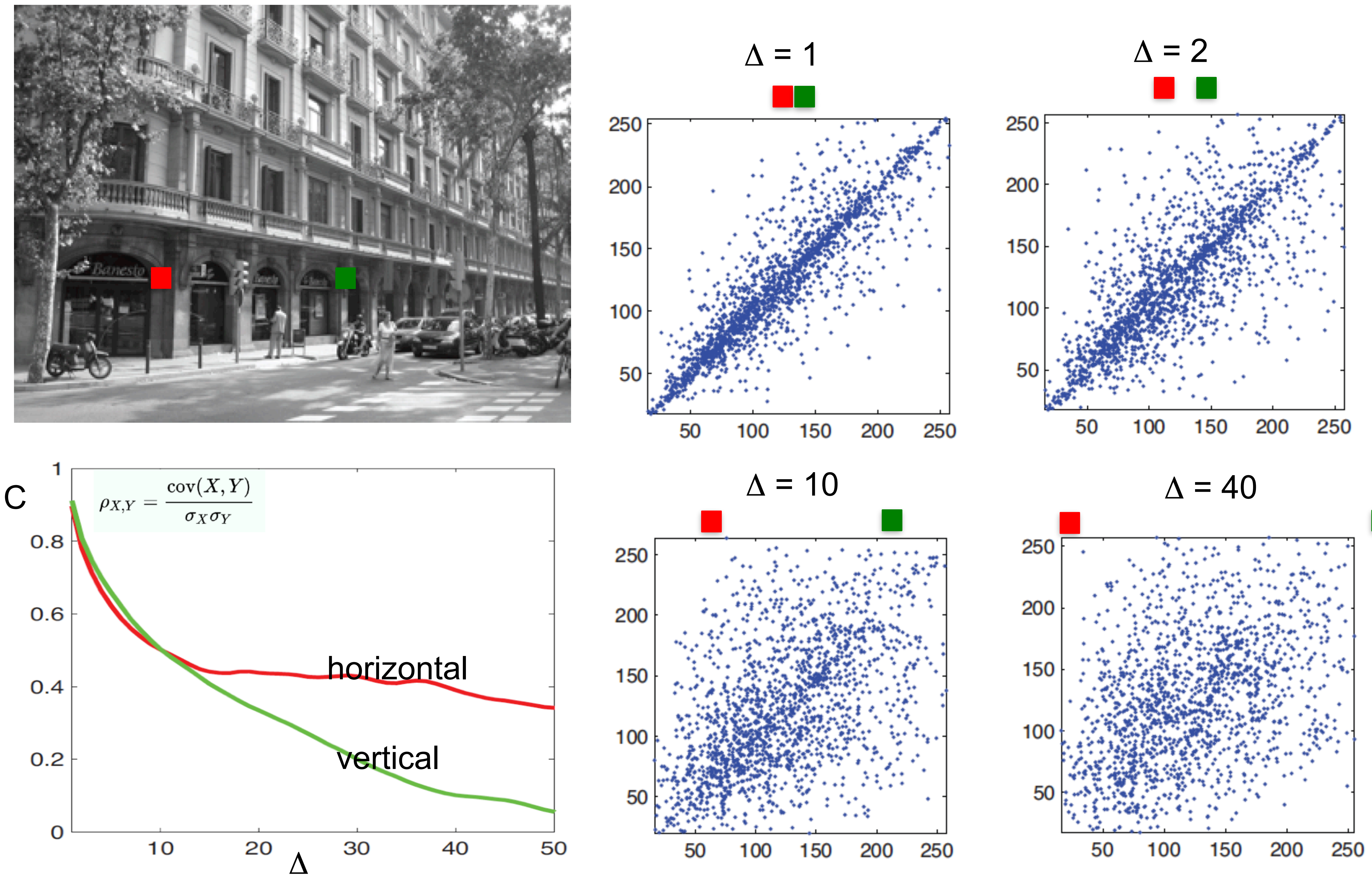


First model: include pixel correlations



$$C(\Delta x, \Delta y) = \mathbf{E}[\mathbf{I}(x + \Delta x, y + \Delta y), \mathbf{I}(x, y)]$$

$$C(\Delta x, \Delta y) = \mathrm{E}\left[\mathbf{I}(x+\Delta x, y+\Delta y), \mathbf{I}(x,y)\right]$$



By the Wiener-Khinchin theorem, the Fourier transform of the auto-correlation function of the image is the power spectrum of the image, so…

A remarkable property of natural images

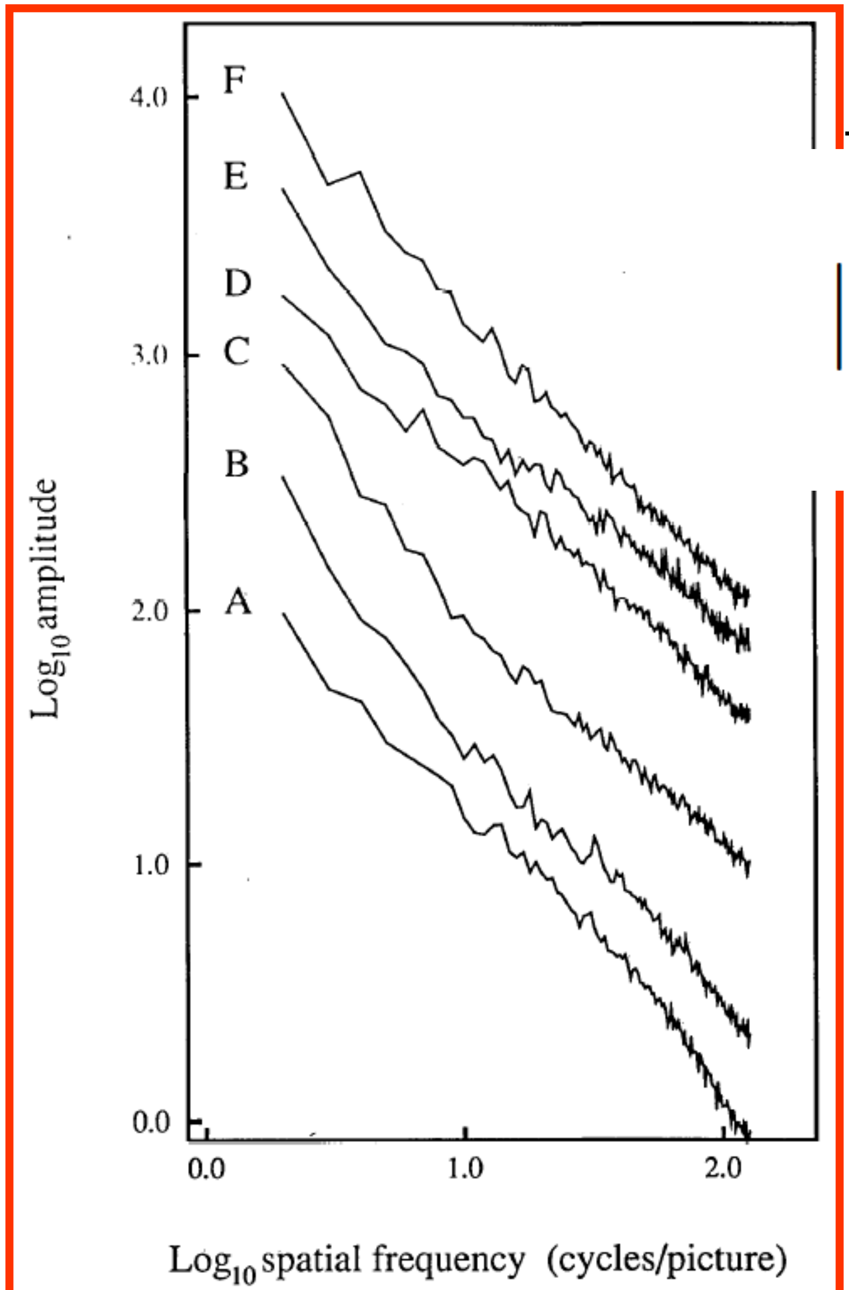
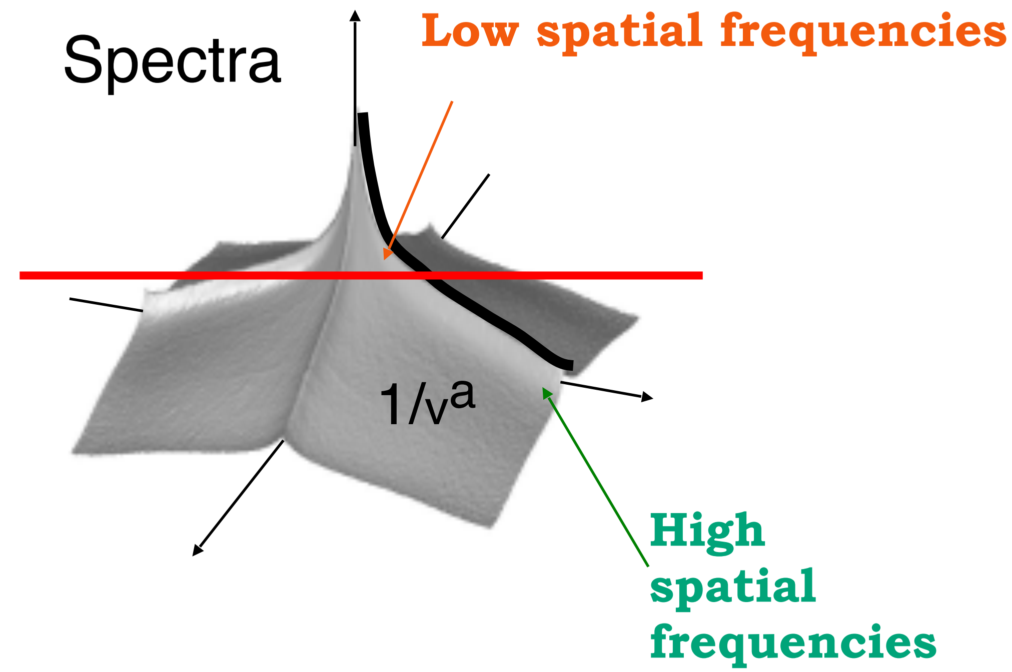
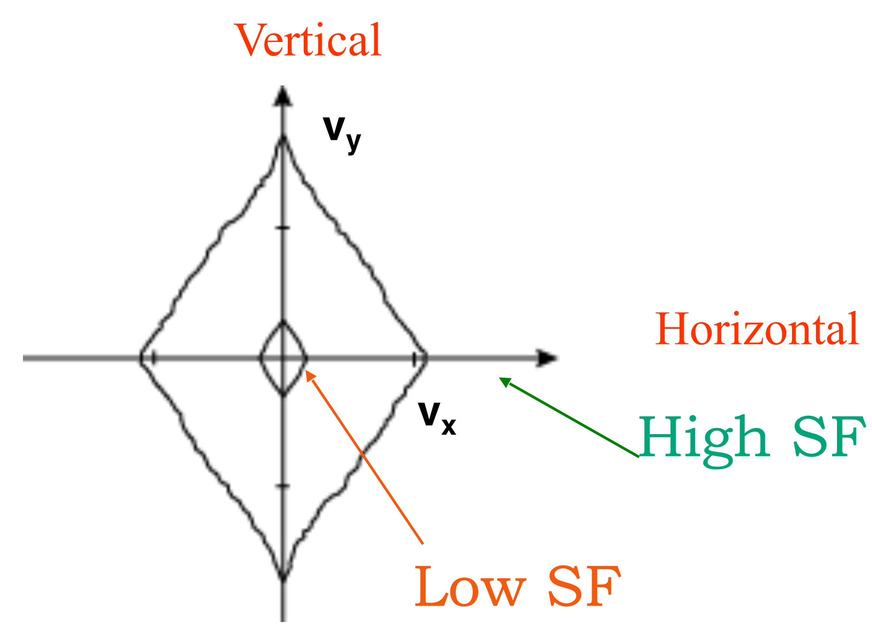


Fig. 8. Amplitude spectra for the six images A–F, averaged across all orientations. The spectra have been shifted up for clarity. On these log–log coordinates the spectra fall off by a factor of roughly 1/f (a slope of −1). Therefore the power spectra fall off as $1/f^2$.

Power spectra fall off as

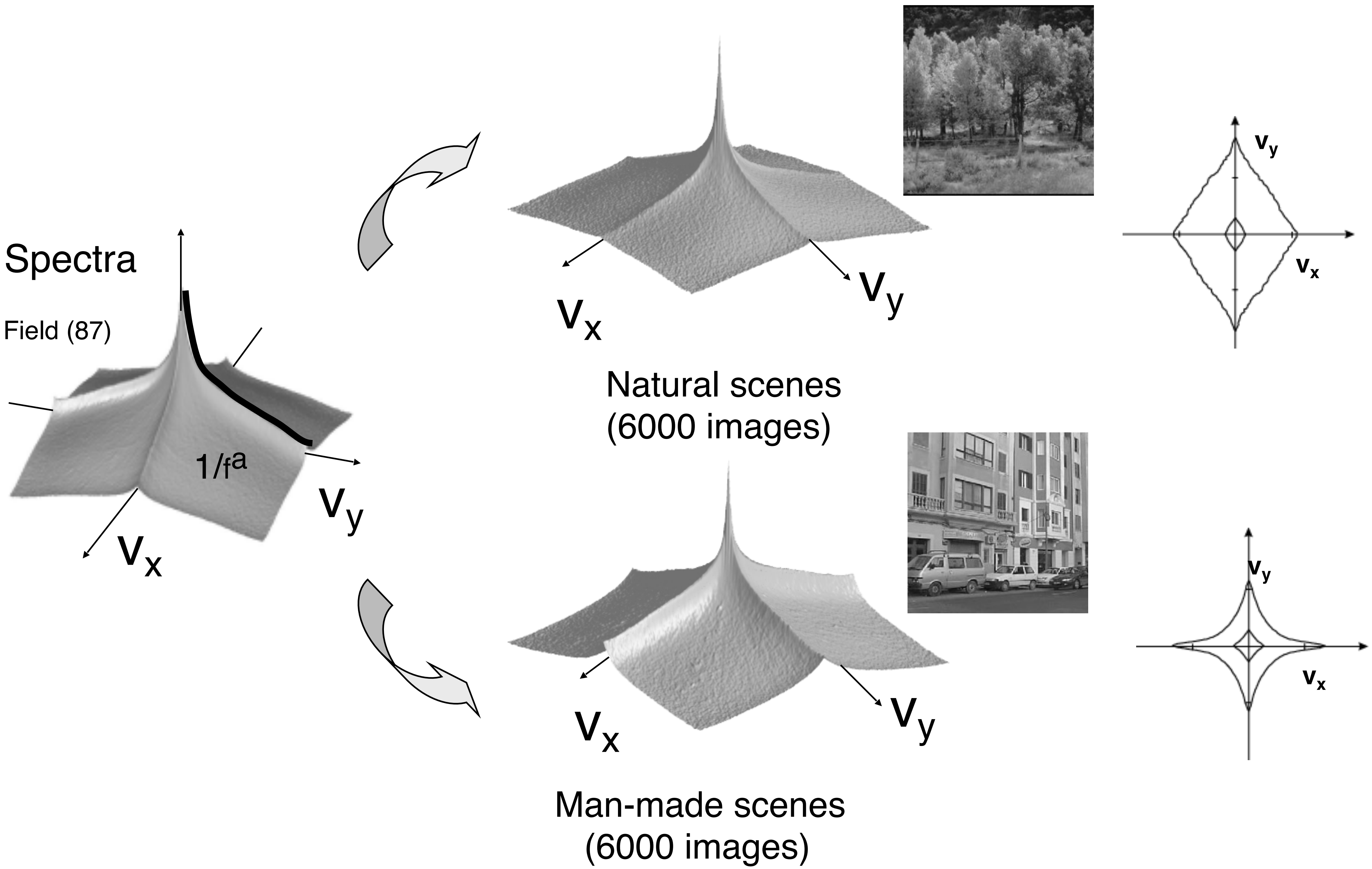
$$|\hat{\mathbf{I}}(v)| \simeq \frac{1}{|v|^{\alpha}}$$





D. J. Field, "Relations between the statistics of natural images and the response properties of cortical cells," J. Opt. Soc. Am. A **4**, 2379- (1987)

A remarkable property of natural images



Torralba and Oliva, *Statistics of Natural Image Categories*. Network: Computation in Neural Systems 14 (2003) 391-412.

Gaussian model

We want a distribution that captures the correlation structure typical of natural images.

Let **C** be the covariance matrix of the image:

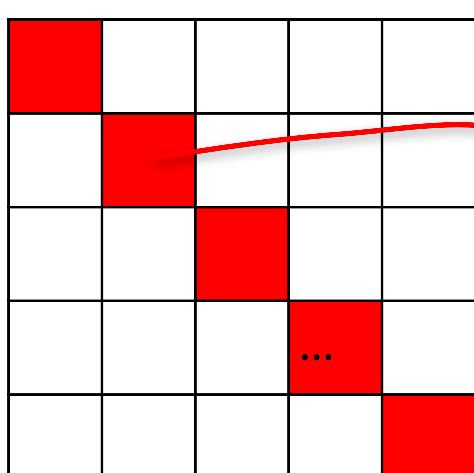
$$p(\mathbf{I}) = \exp\left(-\frac{1}{2}\mathbf{I}^T\mathbf{C}^{-1}\mathbf{I}\right) \qquad C = \begin{bmatrix} c_0 & c_1 & c_2 & & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & & \vdots \\ & c_{n-1} & c_0 & c_1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & c_2 \\ & & & & & c_1 \\ c_1 & \cdots & & c_{n-1} & & c_0 \end{bmatrix}$$

Stationarity assumption: Symmetrical circulant matrix

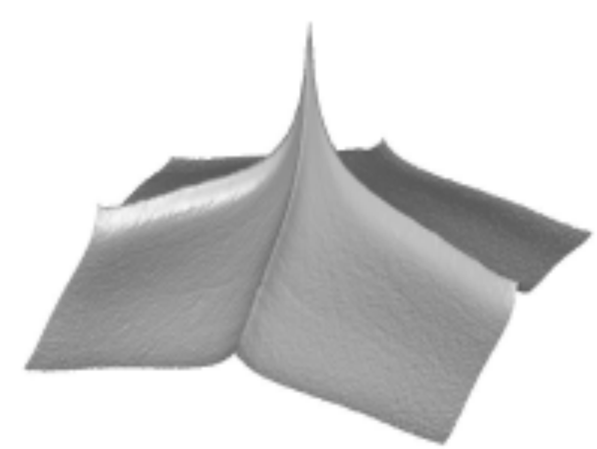
Diagonalization of circulant matrices: $C = EDE^T$

The eigenvectors are the Fourier basis

The eigenvalues are the squared magnitude of the Fourier coefficients

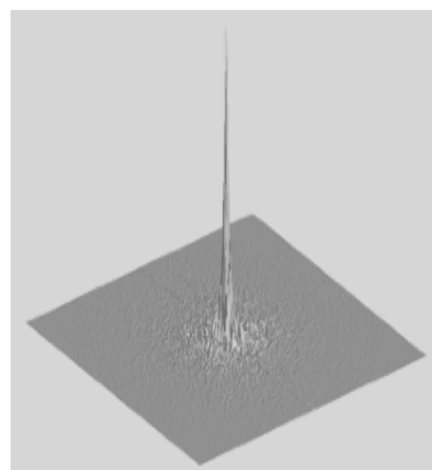
D= 

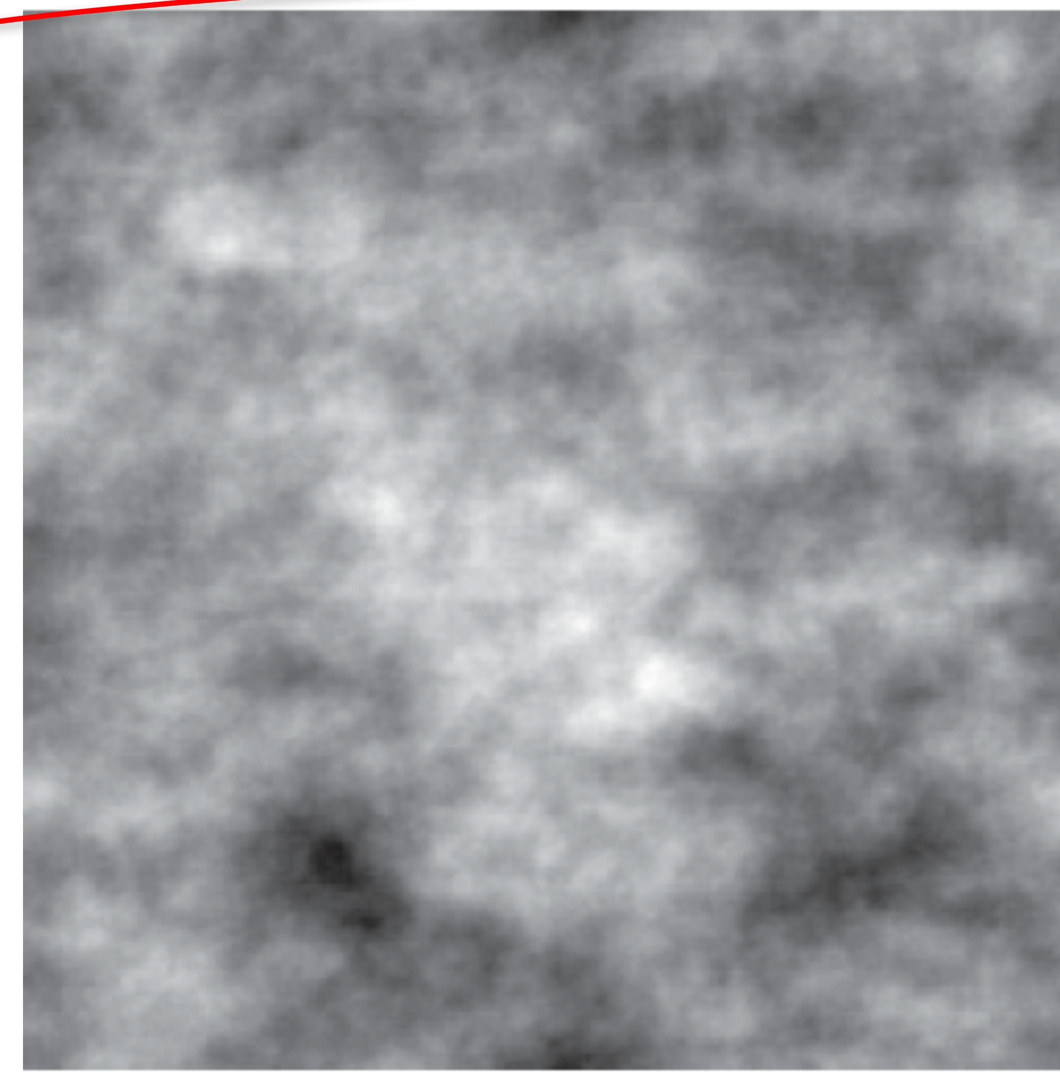
$$\left|\hat{\mathbf{I}}(v)\right|^2 \simeq \frac{1}{|v|^{\alpha}}$$



Sampling new images

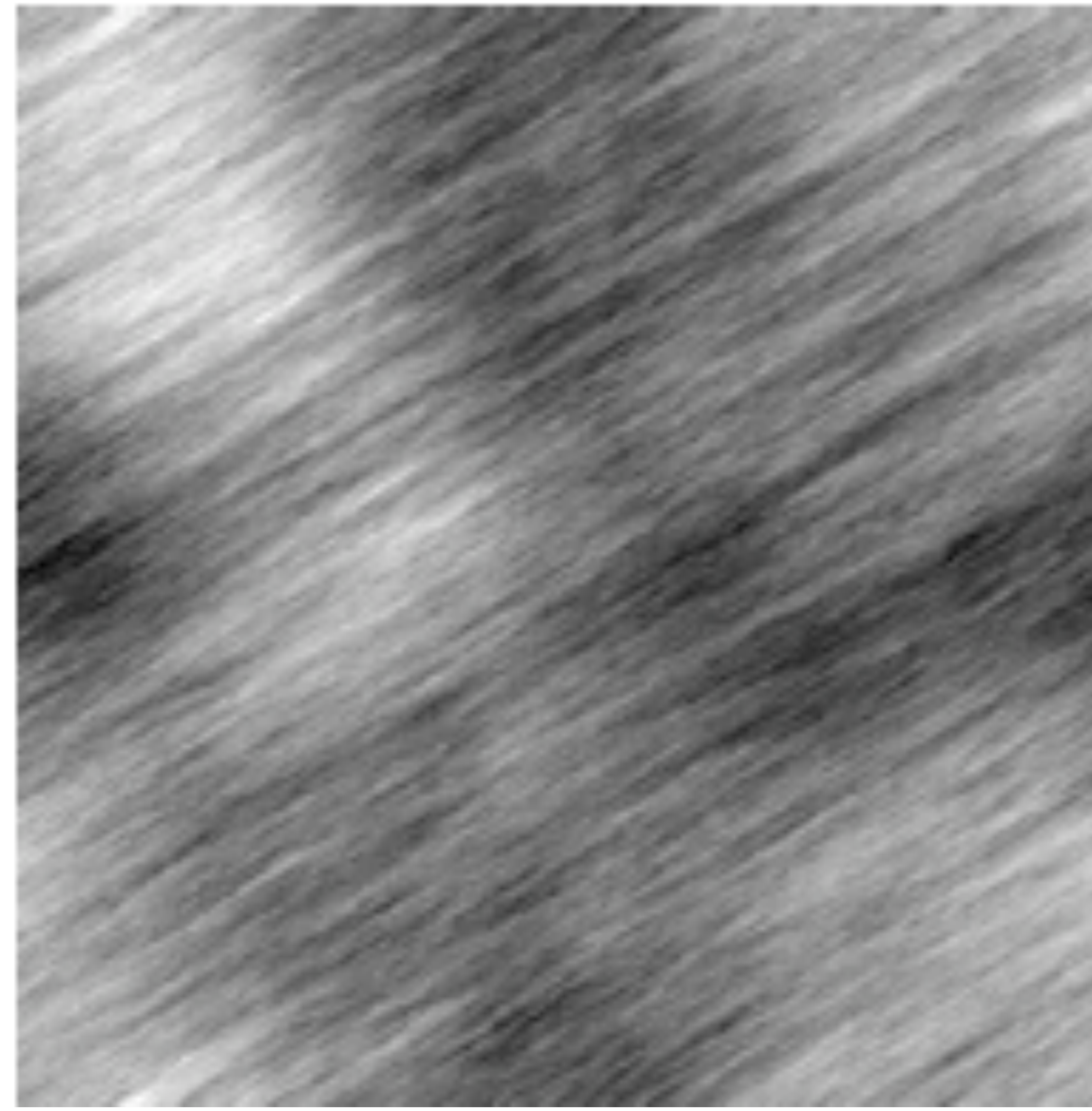
$$p(\mathbf{I}) = \exp\left(-\frac{1}{2}\mathbf{I}^T\mathbf{C}^{-1}\mathbf{I}\right)$$





Sample

Sampling new images



Randomizing the phase (if you fit the Gaussian image model to each of the images in the top row, then draw another random sample, you get the bottom row)

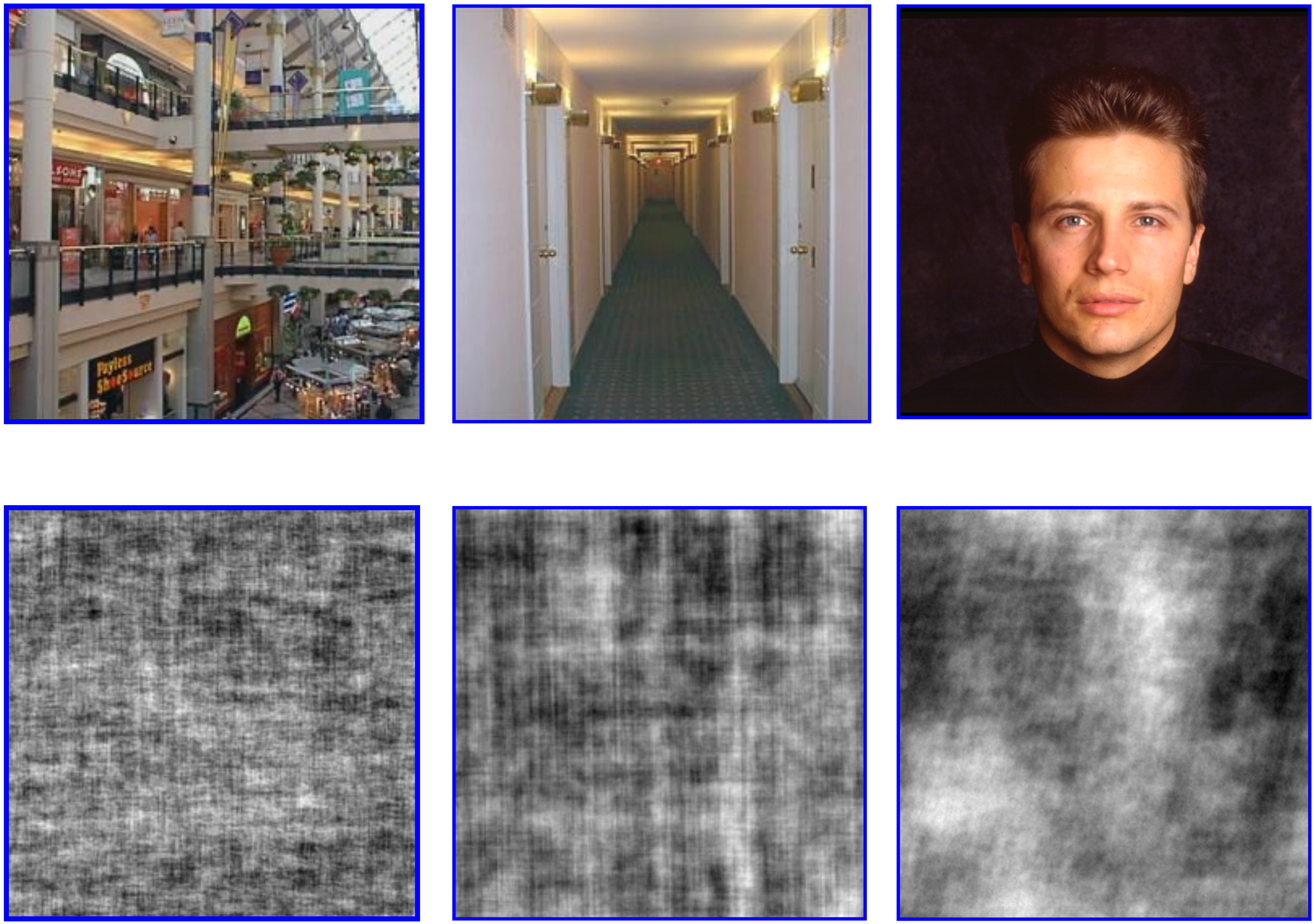
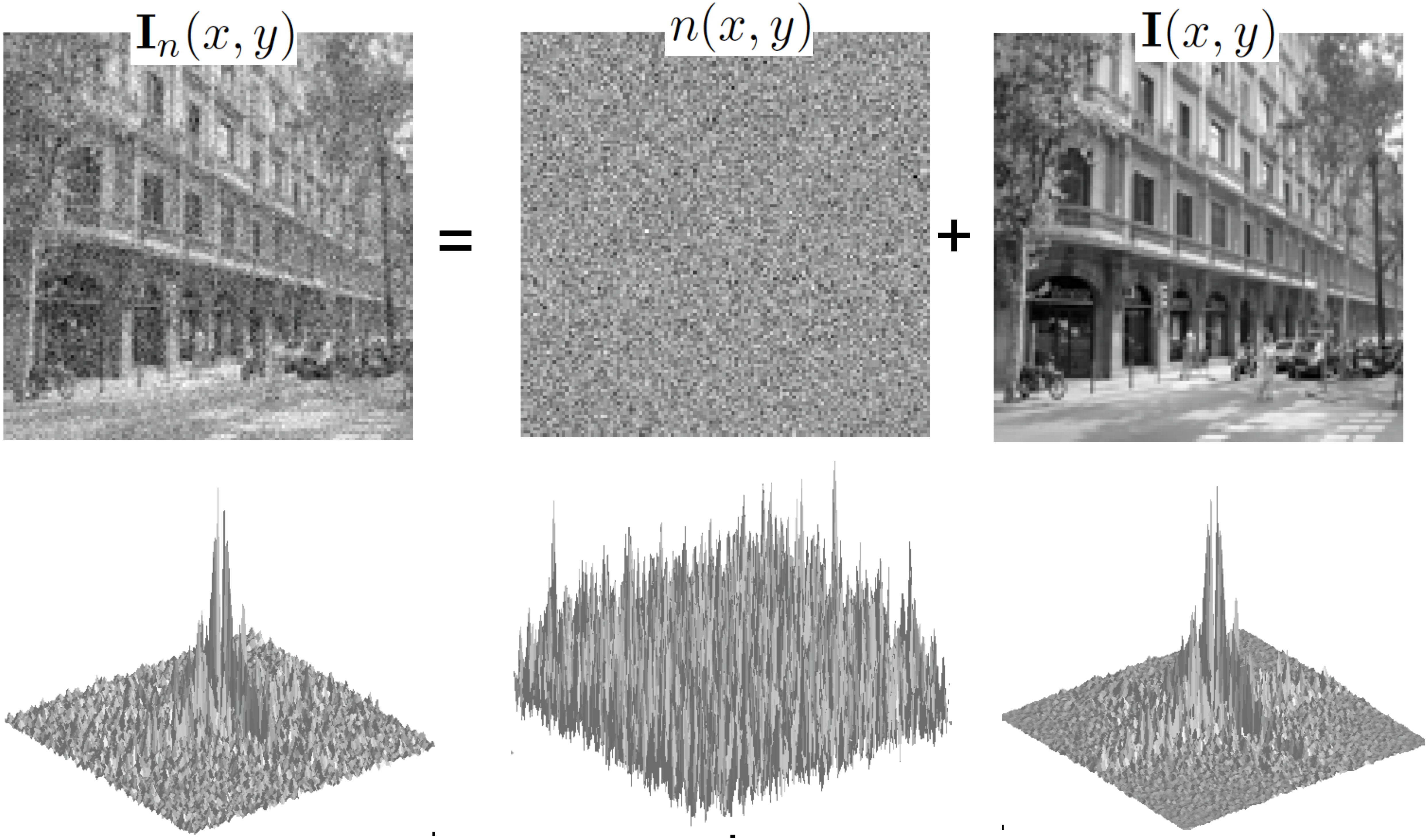


Image model application: Denoising

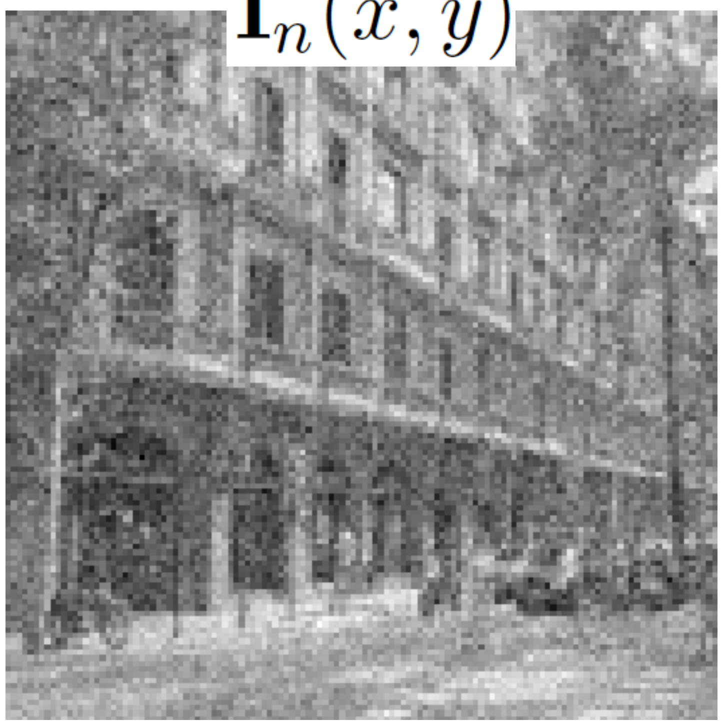
Decomposition of a noisy image



Denoising

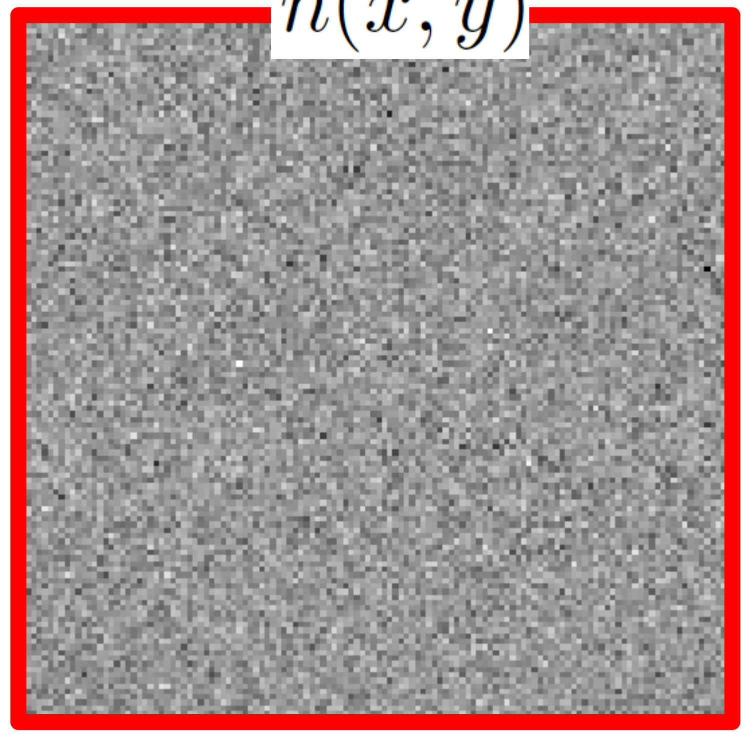
Decomposition of a noisy image

$\mathbf{I}_n(x, y)$



=

$n(x, y)$



+

$\mathbf{I}(x, y)$

White Gaussian noise: $N(0, \sigma_n^2)$

Natural image

Find I(x,y) that maximizes the posterior (maximum a posteriori, MAP):

$$\max_{\mathbf{I}} p(\mathbf{I}|\mathbf{I}_n) = \max_{\mathbf{I}} \quad p(\mathbf{I}_n|\mathbf{I}) \quad \times \quad p(\mathbf{I})$$

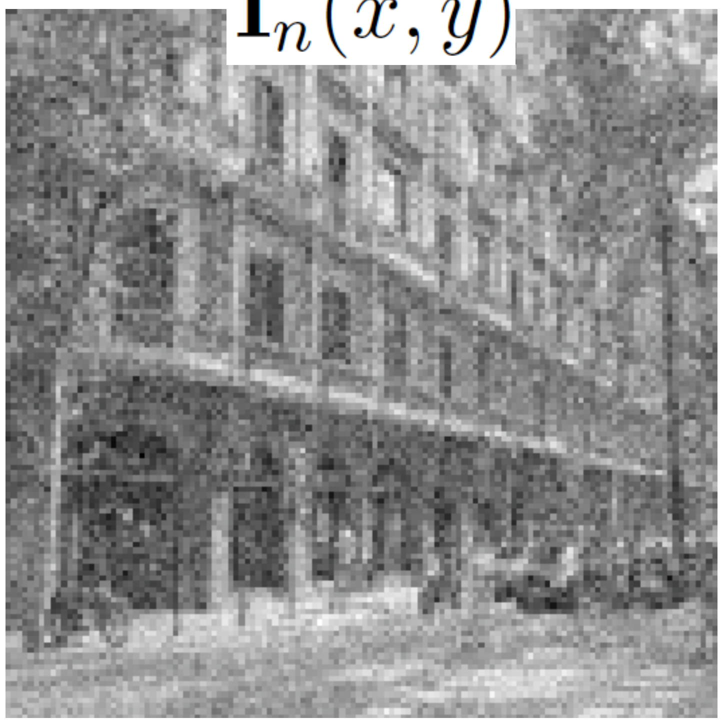
likelihood

prior

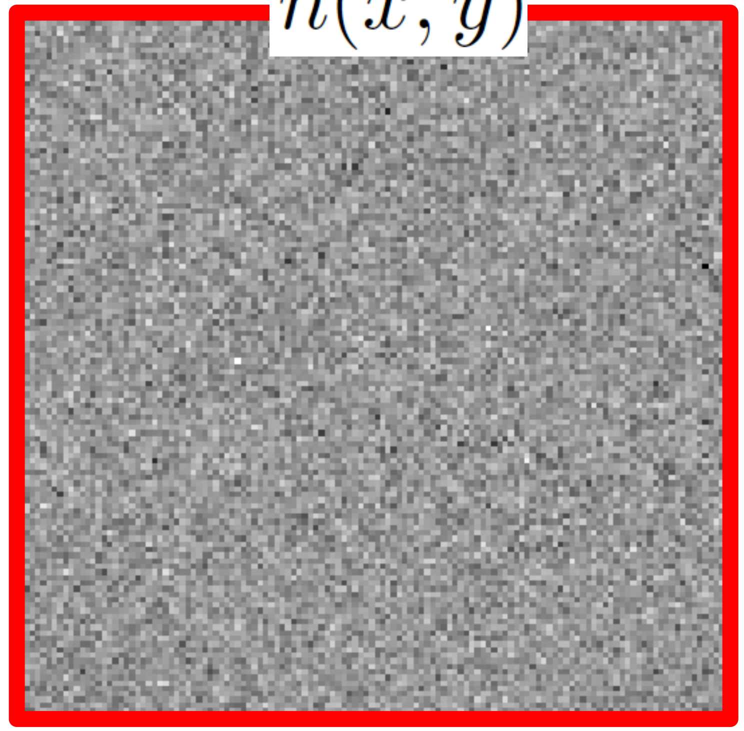
Denoising

Decomposition of a noisy image

$\mathbf{I}_n(x,y)$



$=$ $n(x,y)$



$+$ $\mathbf{I}(x,y)$

White Gaussian noise: $N(0, \sigma_n^2)$

Natural image

Find I(x,y) that maximizes the posterior (maximum a posteriori, MAP):

$$\max_{\mathbf{I}} p(\mathbf{I}|\mathbf{I}_n) = \max_{\mathbf{I}} \underbrace{p(\mathbf{I}_n|\mathbf{I})}_{\text{likelihood}} \times \underbrace{p(\mathbf{I})}_{\text{prior}}$$

$$= \max_{\mathbf{I}} \exp(-\left|\mathbf{I}_n - \mathbf{I}\right|^2 / \sigma_n^2) \times \exp\left(-\frac{1}{2}\mathbf{I}^T\mathbf{C}^{-1}\mathbf{I}\right)$$

Denoising

$$\max_{\mathbf{I}} p(\mathbf{I}|\mathbf{I}_n) = \max_{\mathbf{I}} \quad p(\mathbf{I}_n|\mathbf{I}) \quad \text{x} \quad p(\mathbf{I})$$

likelihood prior

$$= \max_{\mathbf{I}} \exp(-\left|\mathbf{I}_n - \mathbf{I}\right|^2/\sigma_n^2) \text{ x } \exp\left(-\frac{1}{2}\mathbf{I}^T\mathbf{C}^{-1}\mathbf{I}\right)$$

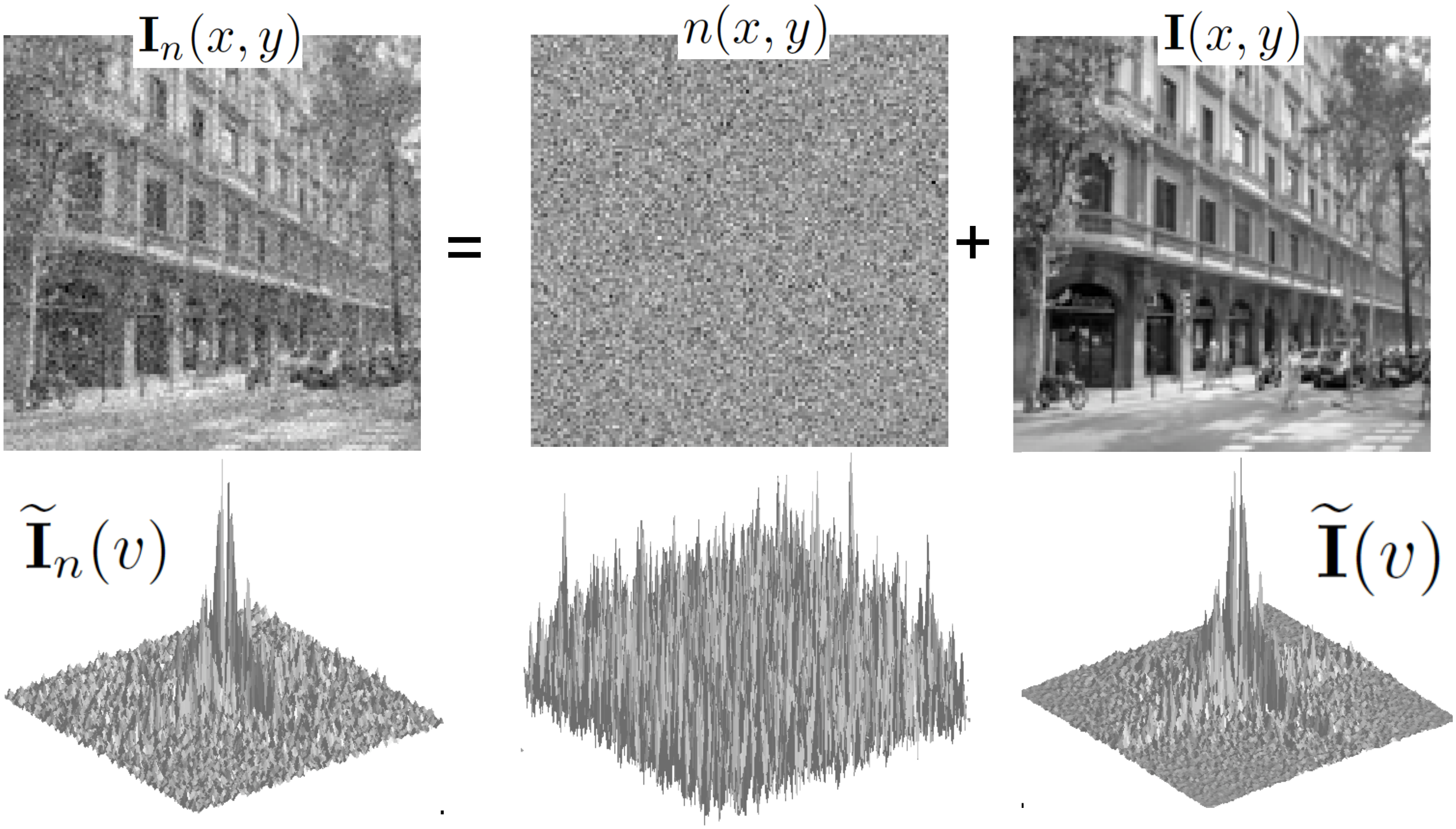
The solution is:

$$\mathbf{I} = \mathbf{C}\left(\mathbf{C} + \sigma_n^2\mathbb{I}\right)^{-1}\mathbf{I}_n$$ (note this is a linear operation)

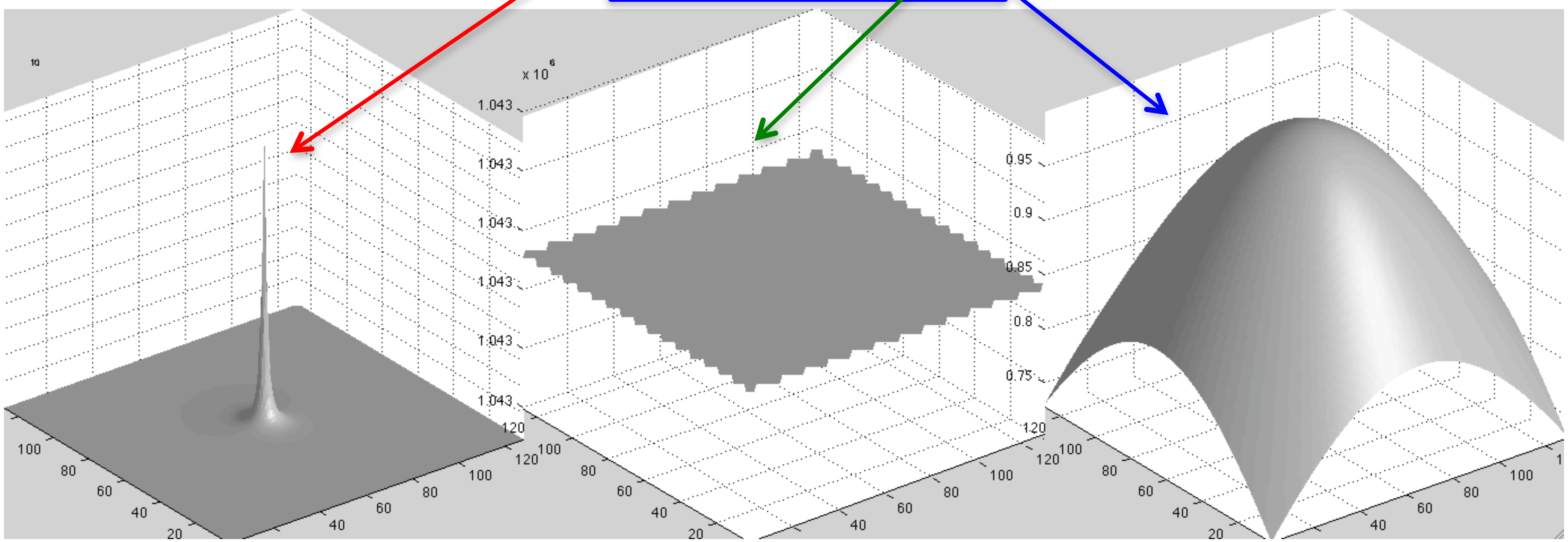
This can also be written in the Fourier domain, with C = EDE^T:

$$\widetilde{\mathbf{I}}(v) = \frac{A/|v|^{2\alpha}}{A/|v|^{2\alpha} + \sigma_n^2}\widetilde{\mathbf{I}}_n(v)$$

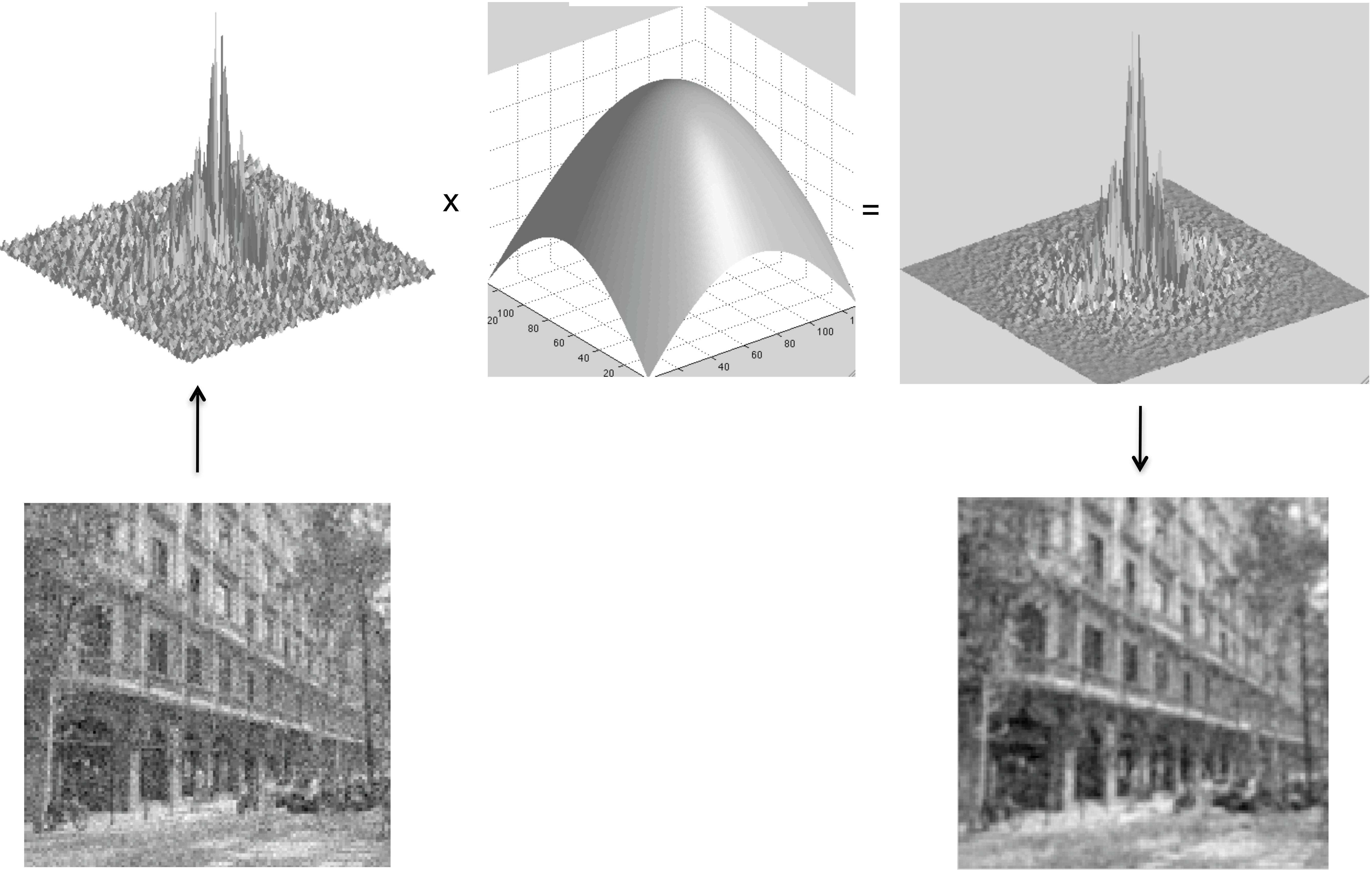
Decomposition of a noisy image



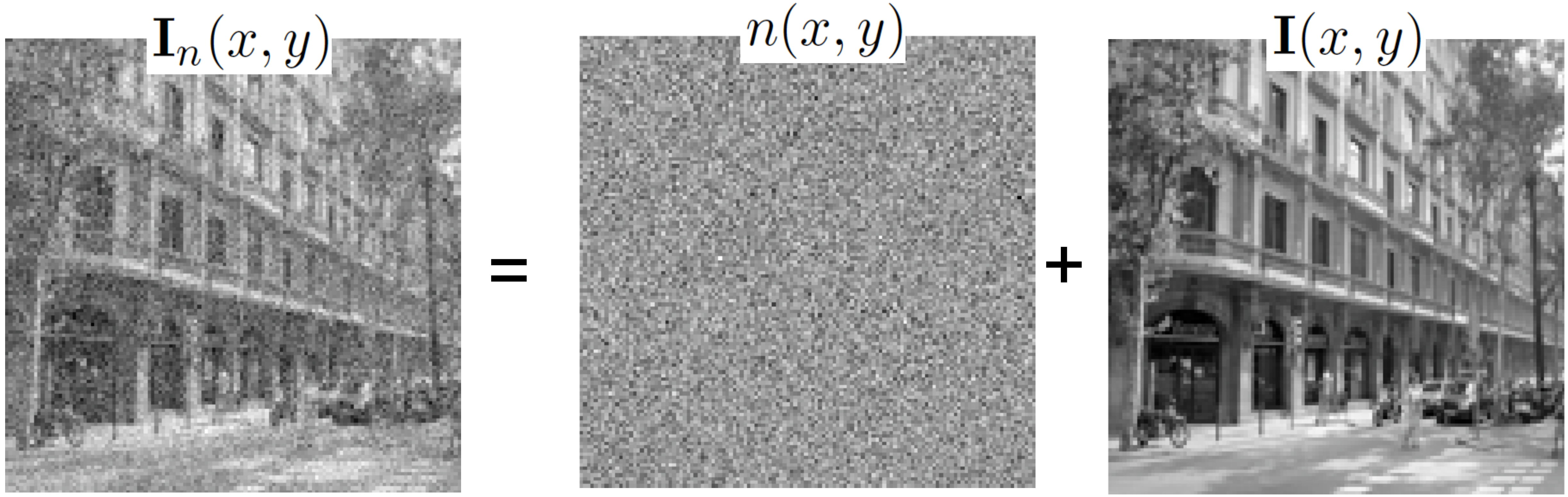
$$\widetilde{\mathbf{I}}(v) = \frac{A/|v|^{2\alpha}}{A/|v|^{2\alpha} + \sigma_n^2}\widetilde{\mathbf{I}}_n(v)$$



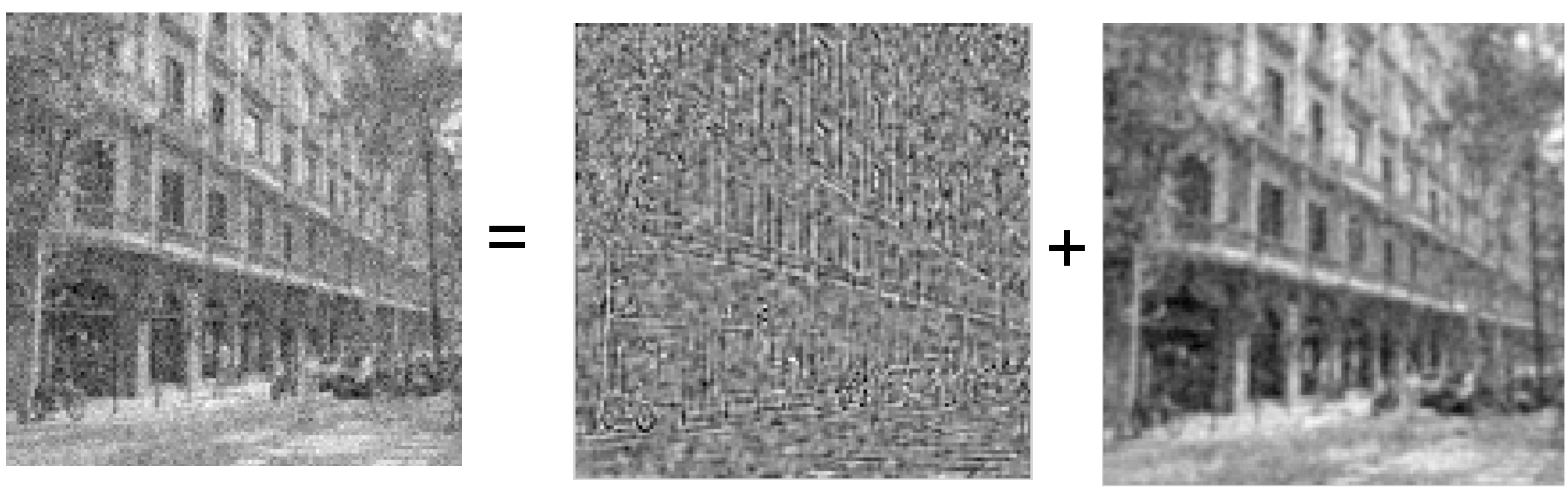
$$\frac{A/|v|^{2\alpha}}{A/|v|^{2\alpha} + \sigma_n^2}$$



The truth:



The estimated decomposition:

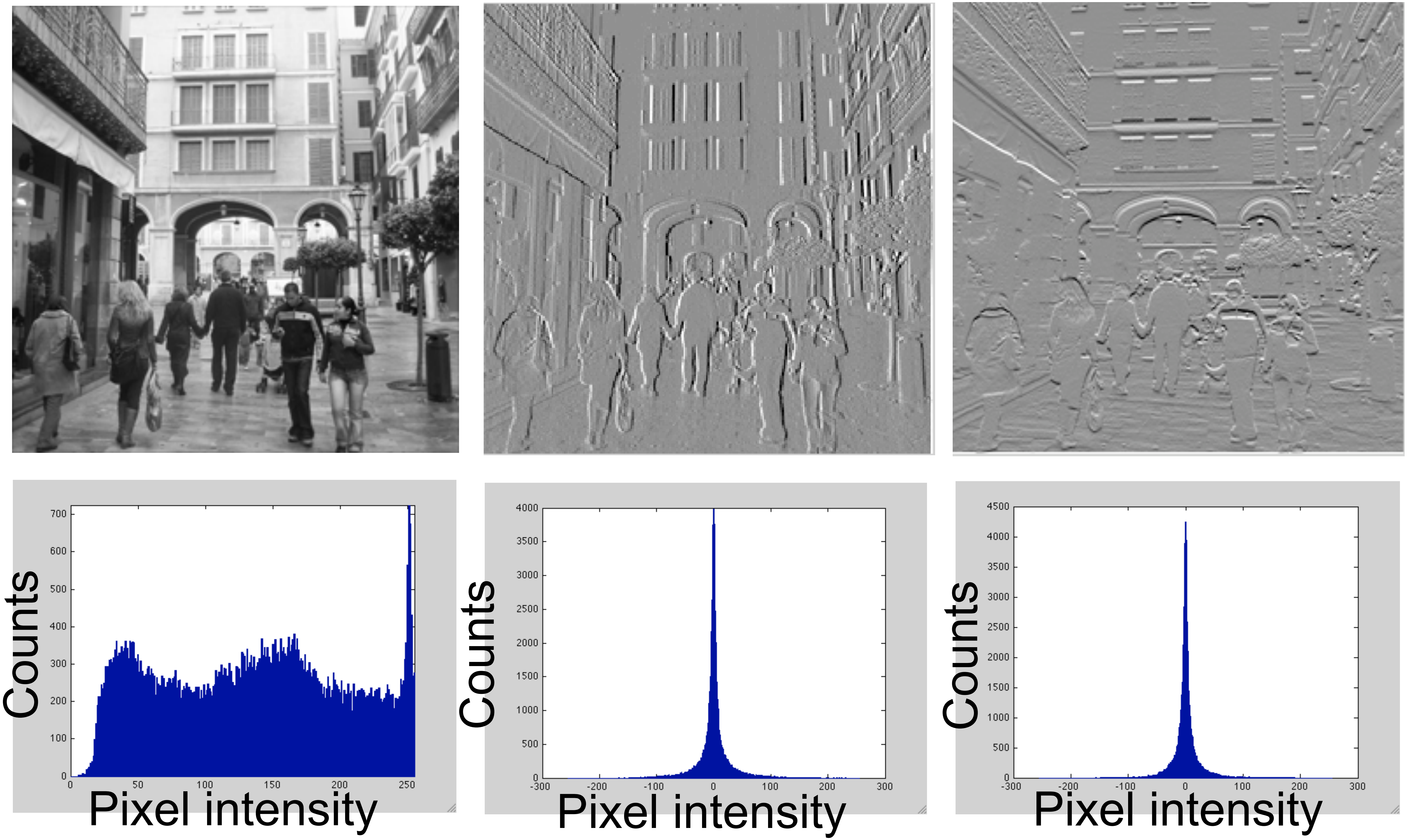


And we got all this from just modeling the correlation between pairs of pixels!

Statistical Image Models

- Gaussian image model
  - image synthesis
  - Wiener filter denoising
- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising
- Non-parametric model
  - image synthesis (Efros and Leung texture model)
  - Non-local means denoising

Observation: Sparse filter response



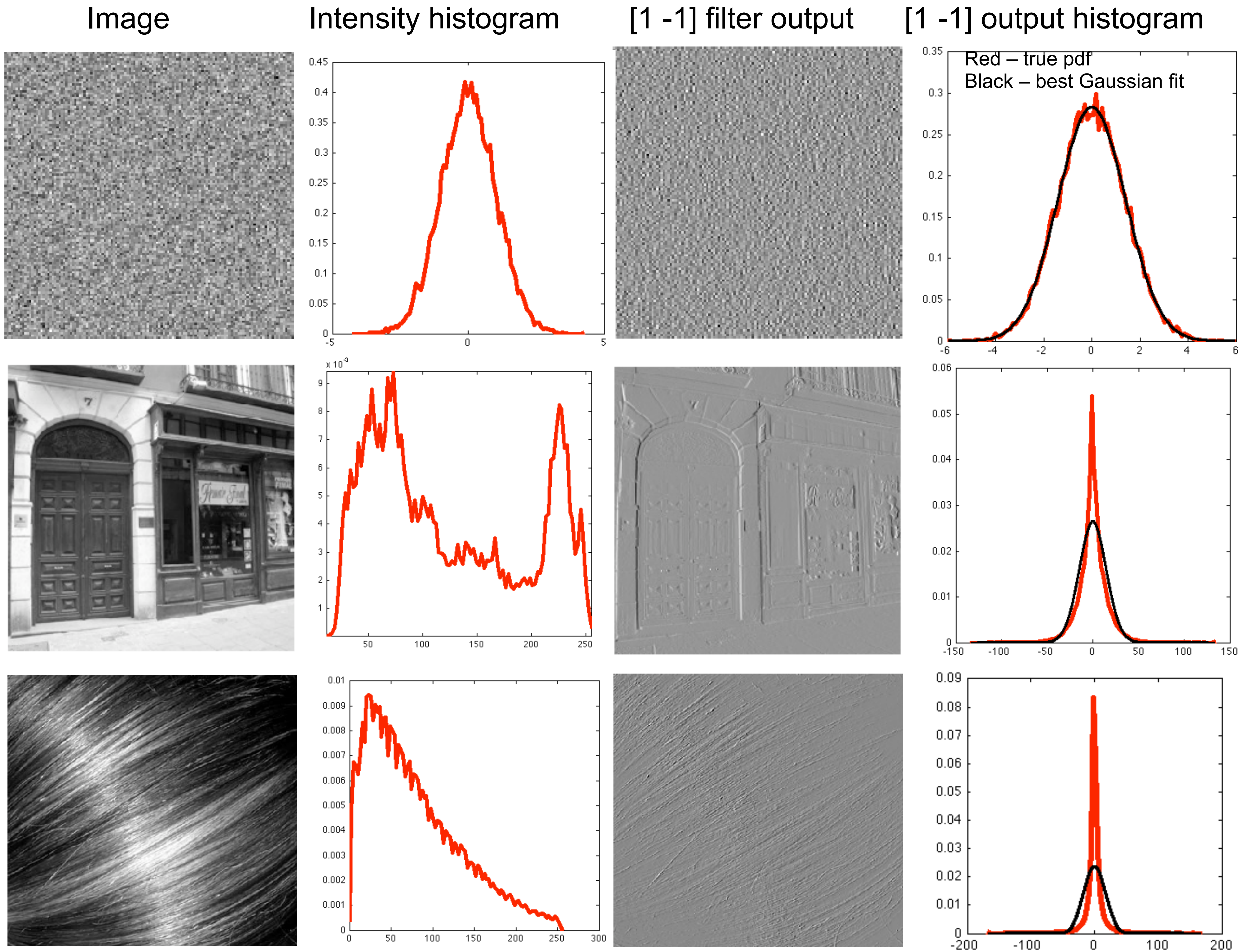
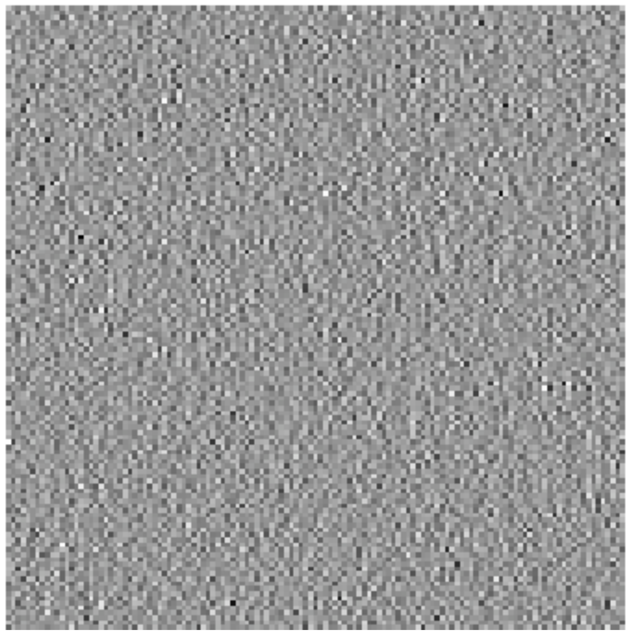
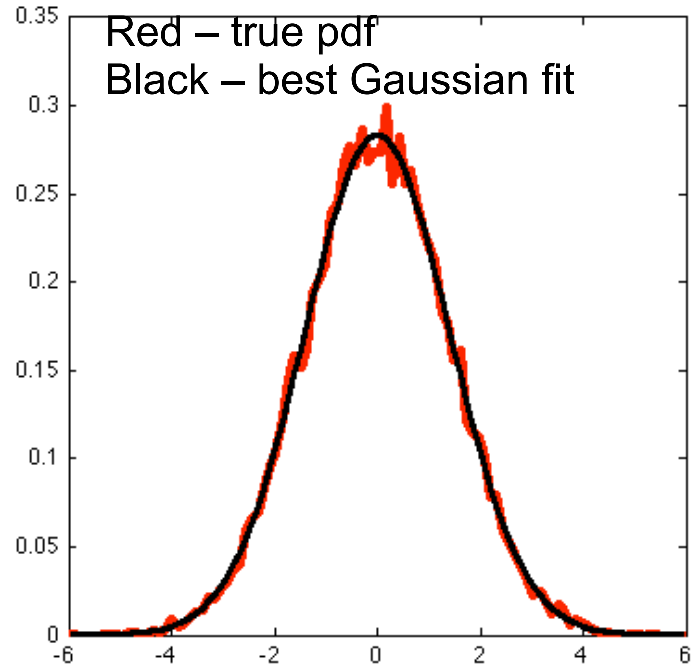


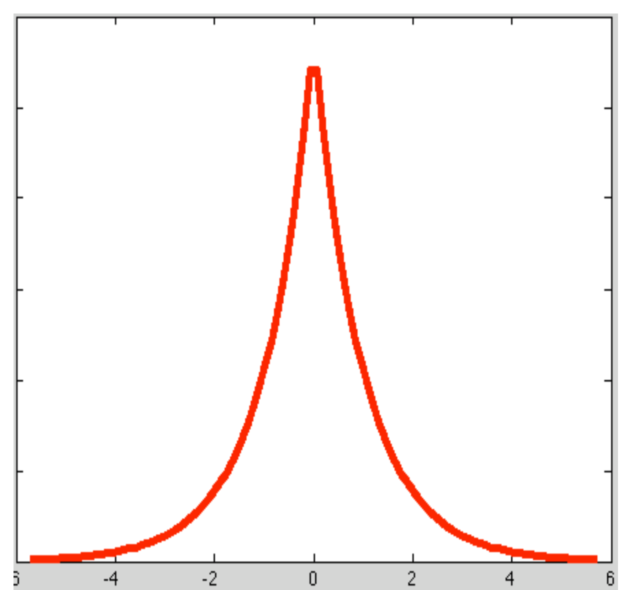
Image
Intensity histogram
[1 -1] filter output
[1 -1] output histogram
Red – true pdf
Black – best Gaussian fit

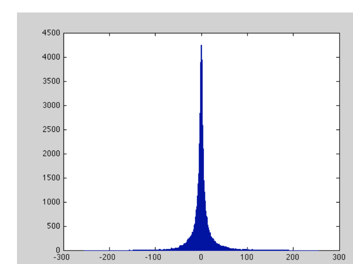
A model for the distribution of filter outputs





$$p(x) = \frac{\exp(-x^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}$$





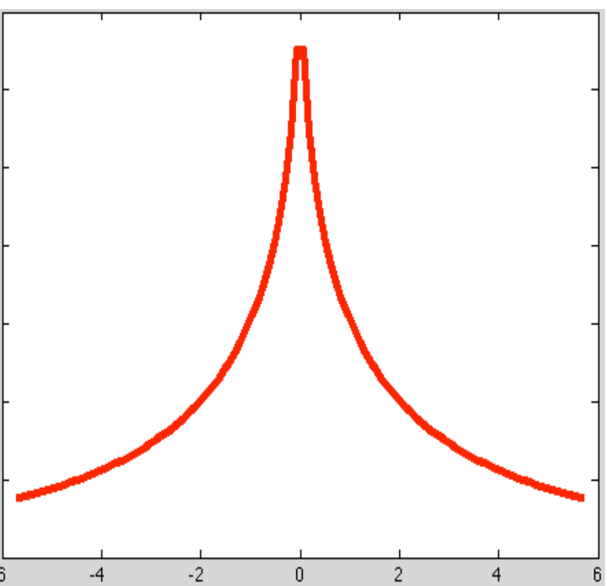
$$p(x) = \frac{\exp(-|x/s|^r)}{2s/r\Gamma(1/r)}$$

$r \sim 0.8$ (< 2)

Generalized Gaussian

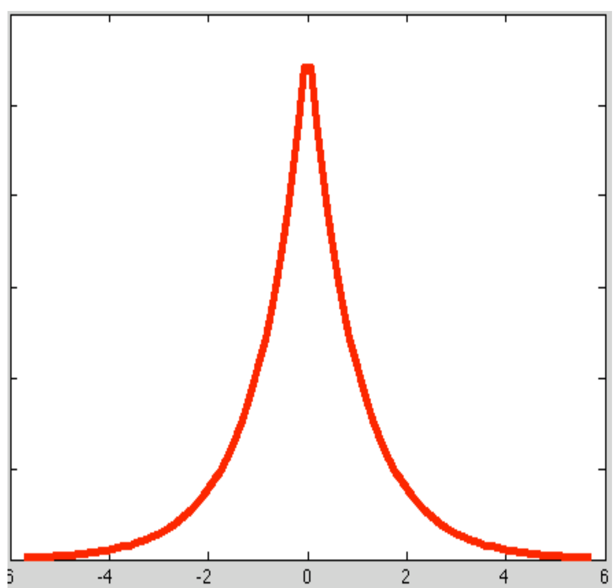
$$p(x) = \frac{\exp(-|x/s|^r)}{2s/r\Gamma(1/r)}$$

r = 0.5



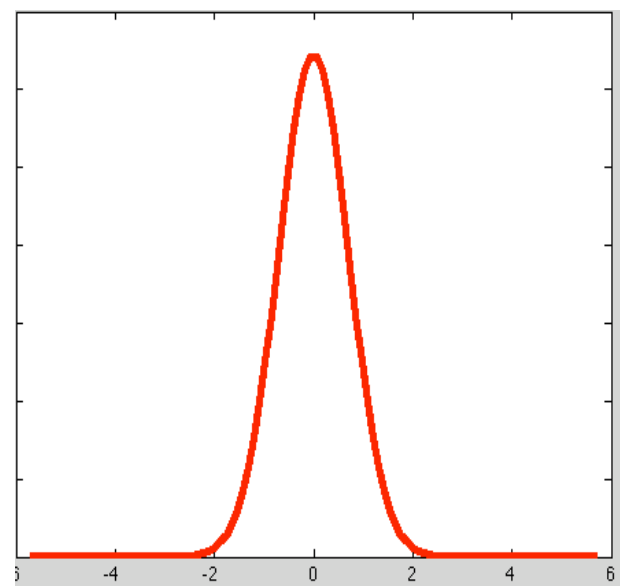
r = 1

Laplacian distribution

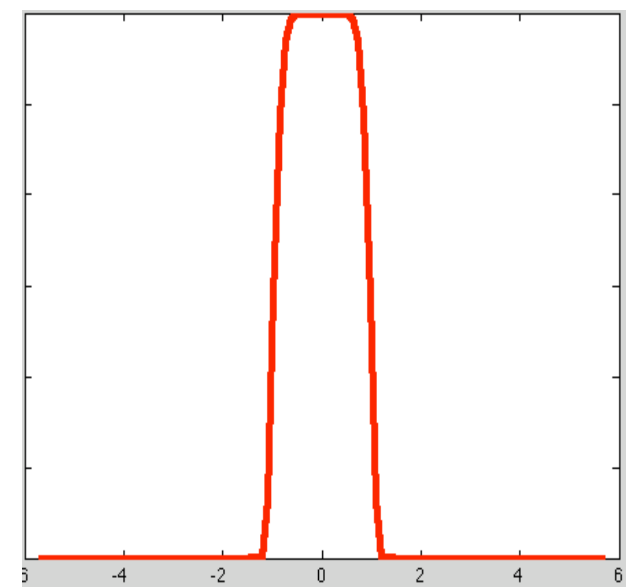


r = 2

Gaussian distribution



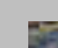
r = 10



Uniform distribution
r -> infinite

The wavelet marginal model

A small neighborhood

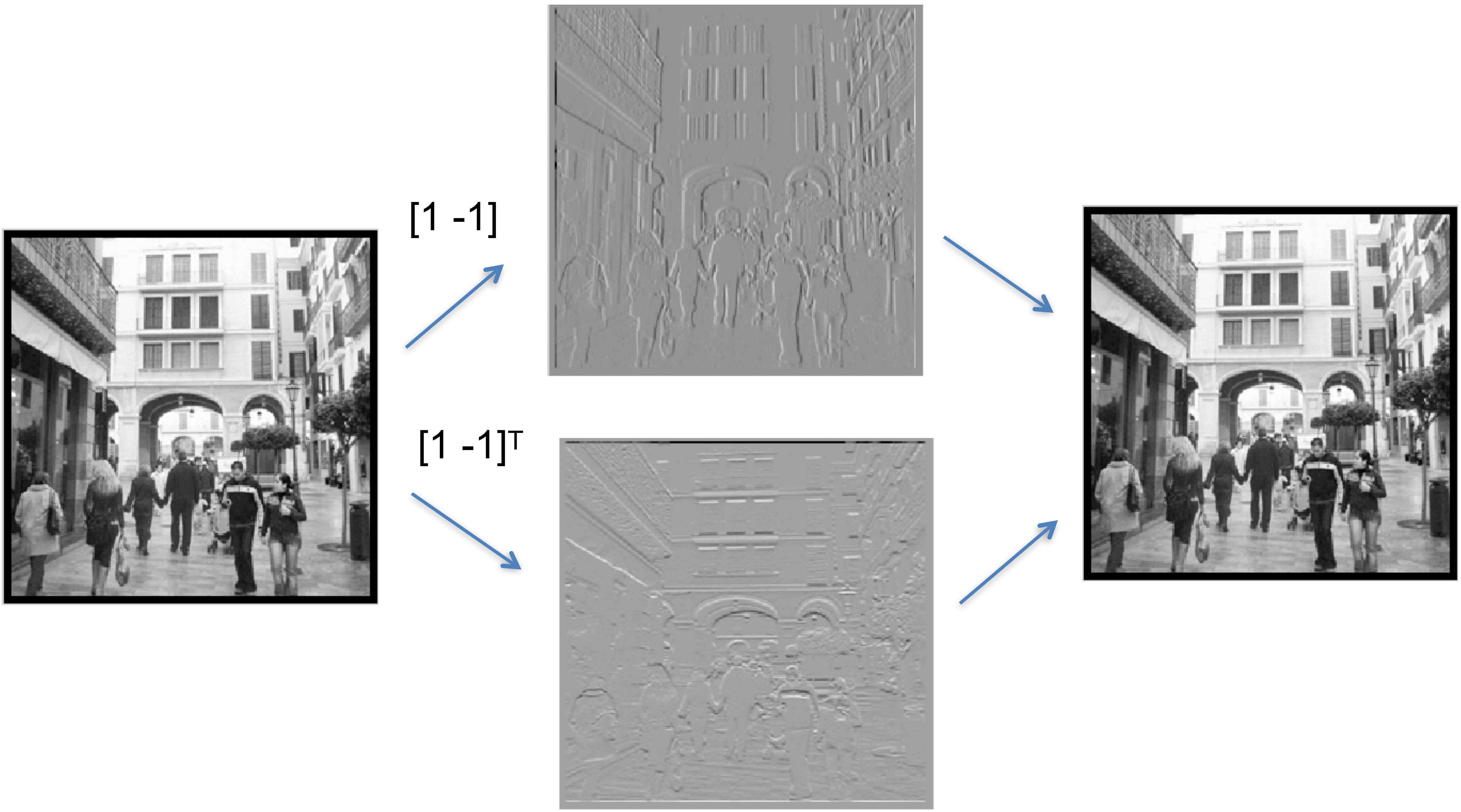


$$p(\mathbf{I}) = \prod_{k} \prod_{x,y} p(h_k(x,y))$$

All pixels and all outputs are independent

Filter outputs

The wavelet marginal model



$$p(\mathbf{I}) = \prod_{k} \prod_{x,y} \mathrm{p}(\mathrm{h}_k(x,y))$$

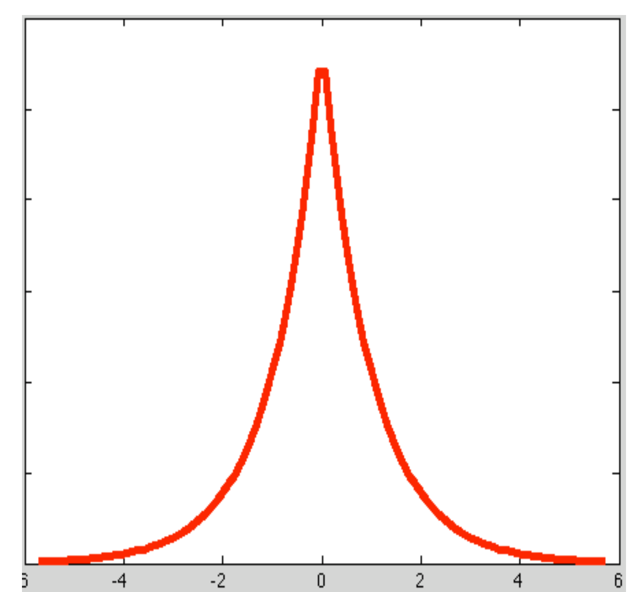
What is the most probable image under the wavelet marginal model?

$[1\ -1]$

$[1\ -1]^T$

$$p(\mathbf{I}) = \prod_{k} \prod_{x,y} p(h_k(x,y))$$

$$p(x) = \frac{\exp(-|x/s|^r)}{2s/r\Gamma(1/r)}$$



Sampling images

Gaussian model

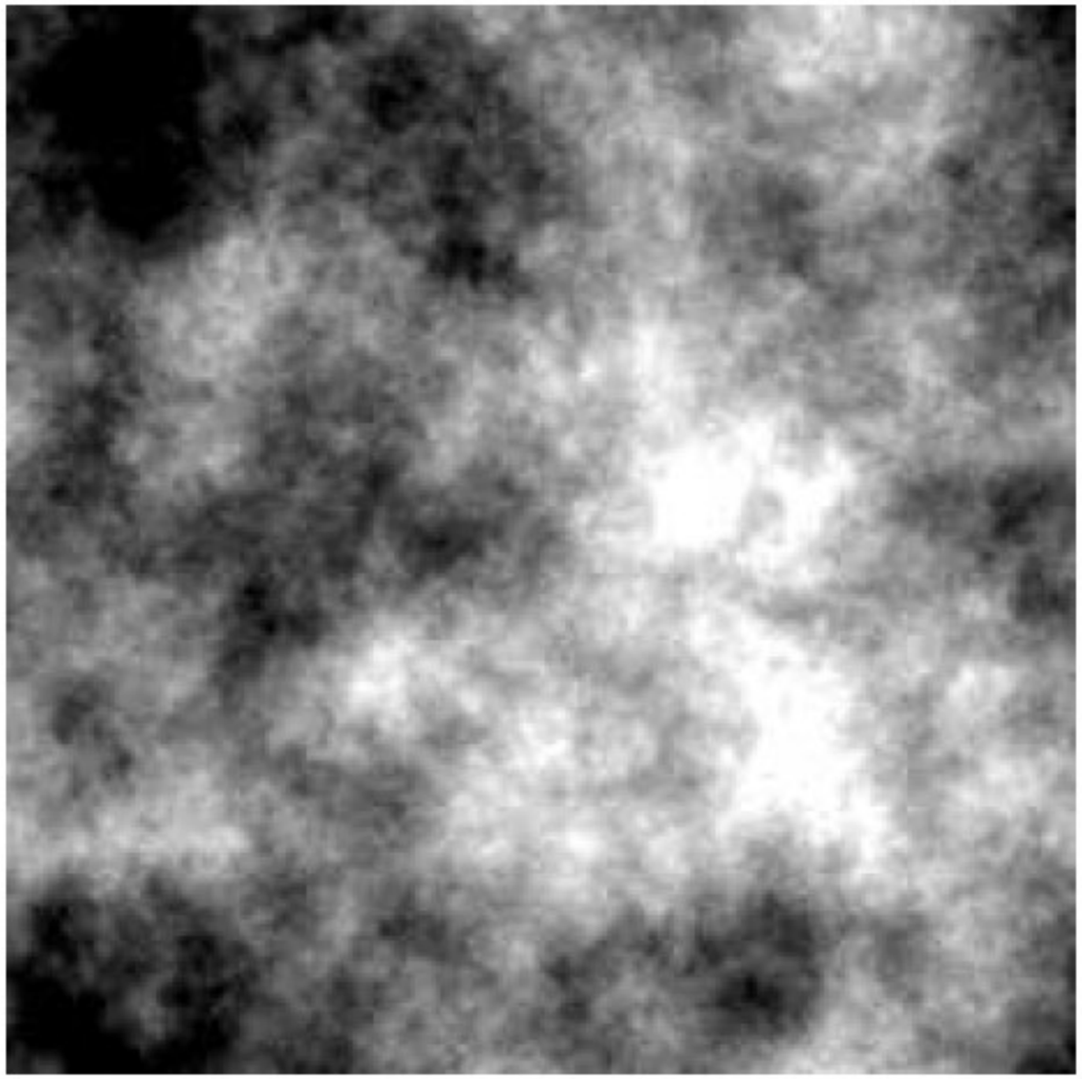


Fig. 3. Example image randomly drawn from the Gaussian spectral model, with $\gamma = 2.0$.

Wavelet marginal model

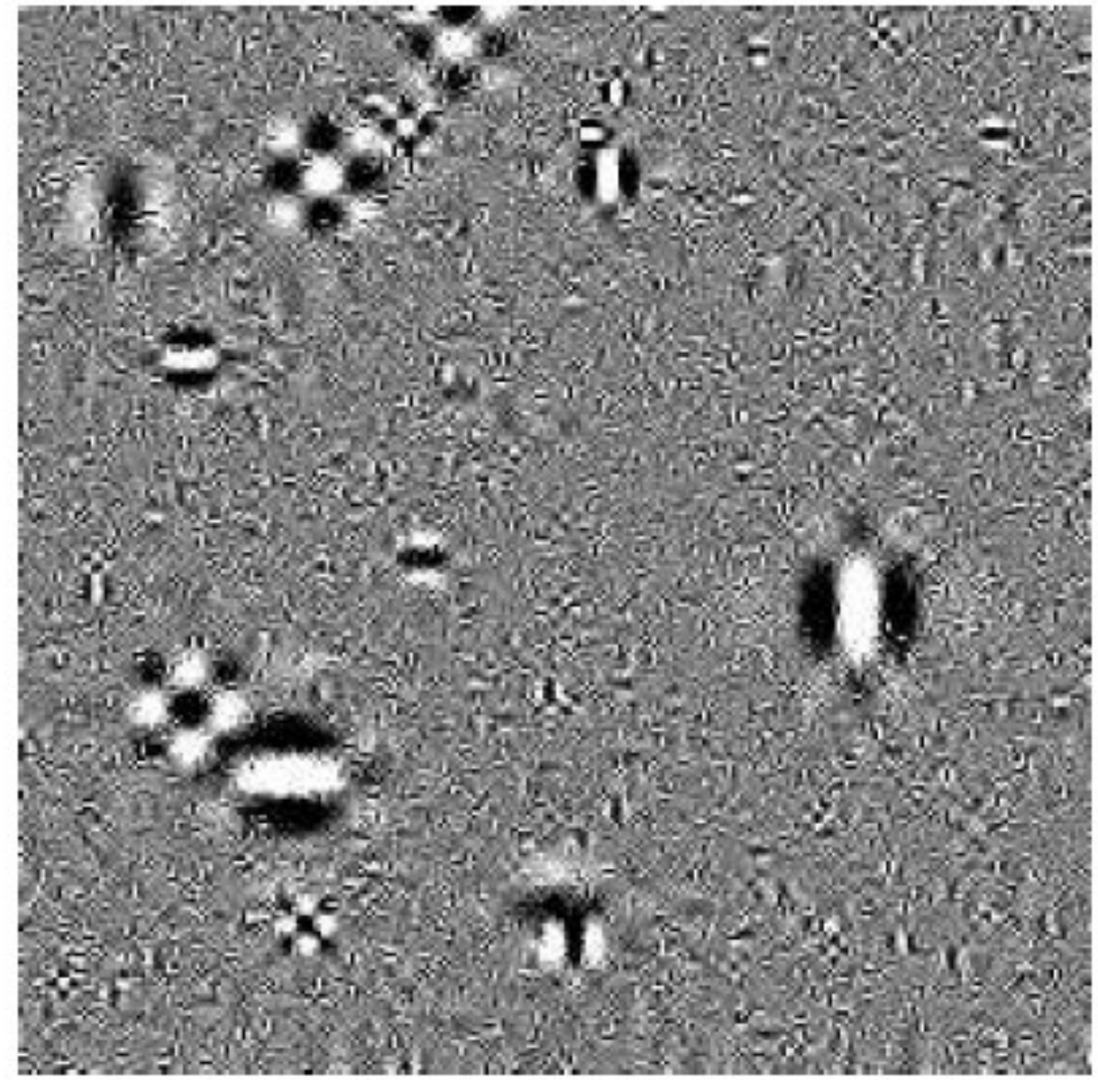
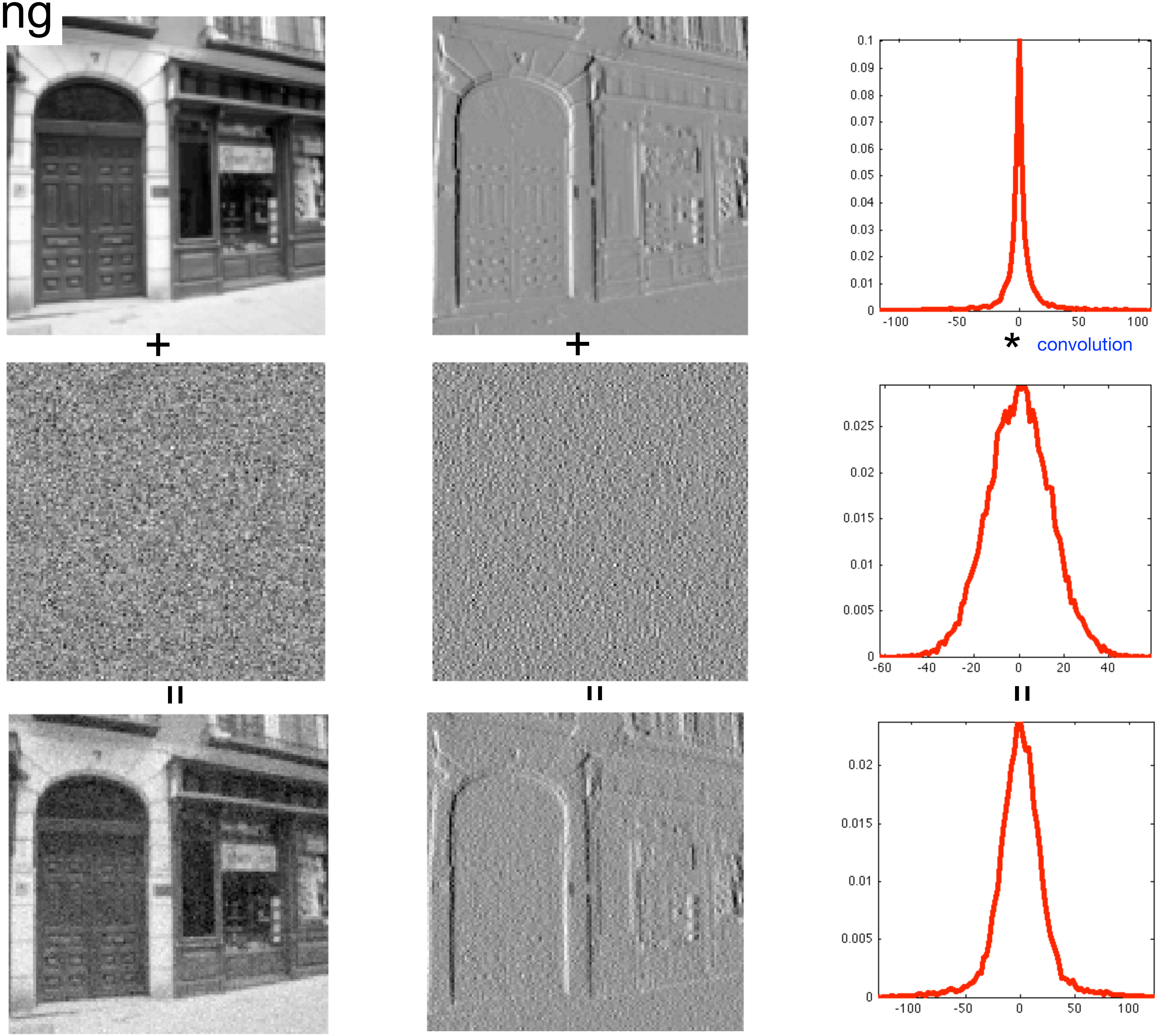


Fig. 6. A sample image drawn from the wavelet marginal model, with subband density parameters chosen to fit the image of Fig. 7.

Denoising



White Gaussian noise

Noisy image

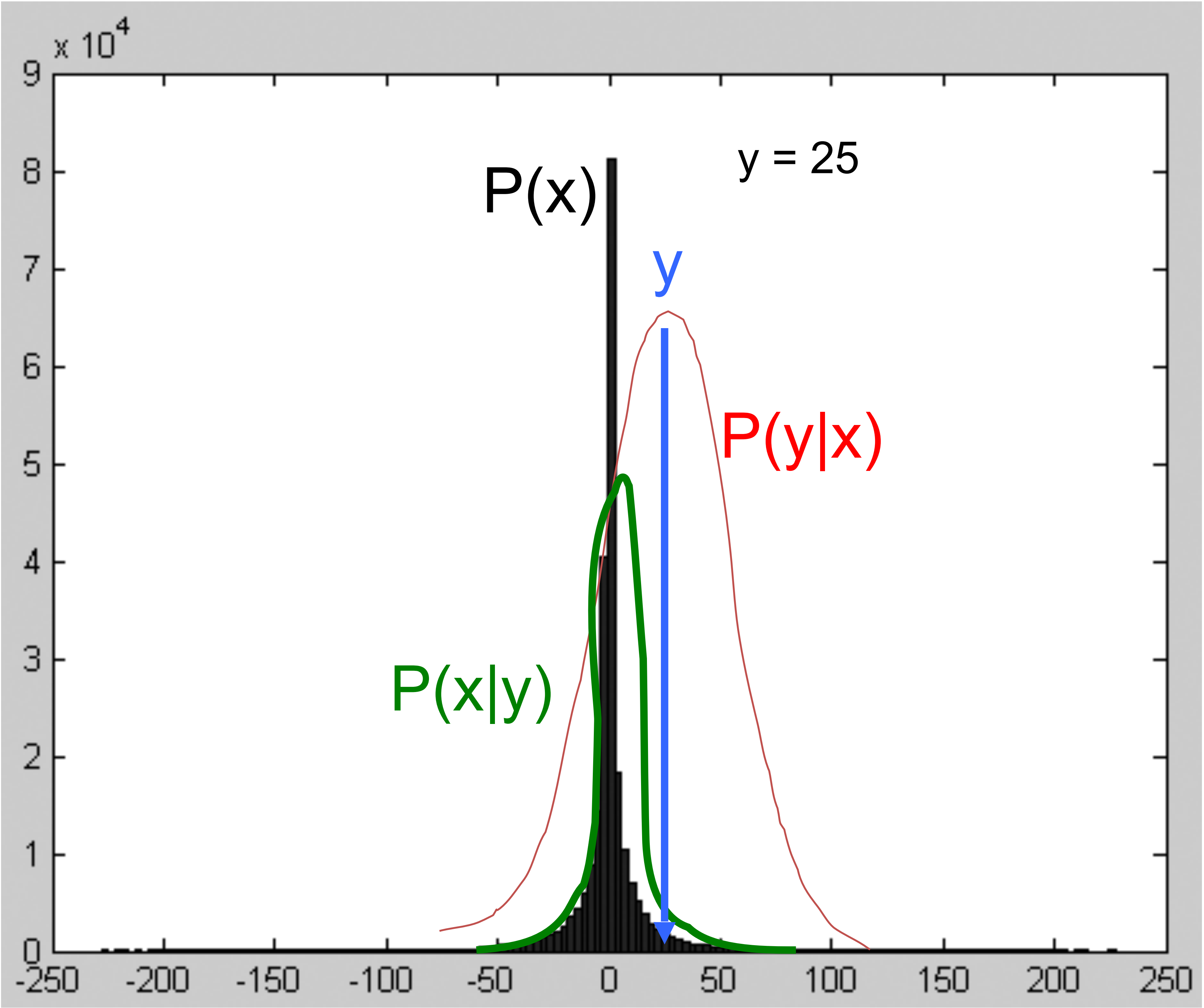
Denoising with the marginal wavelet model

Let y = noise-corrupted observation: y = x+n, with n ~ gaussian.

Let x = bandpassed image value before adding noise.

By Bayes theorem

P(x|y) ~ P(y|x) P(x)

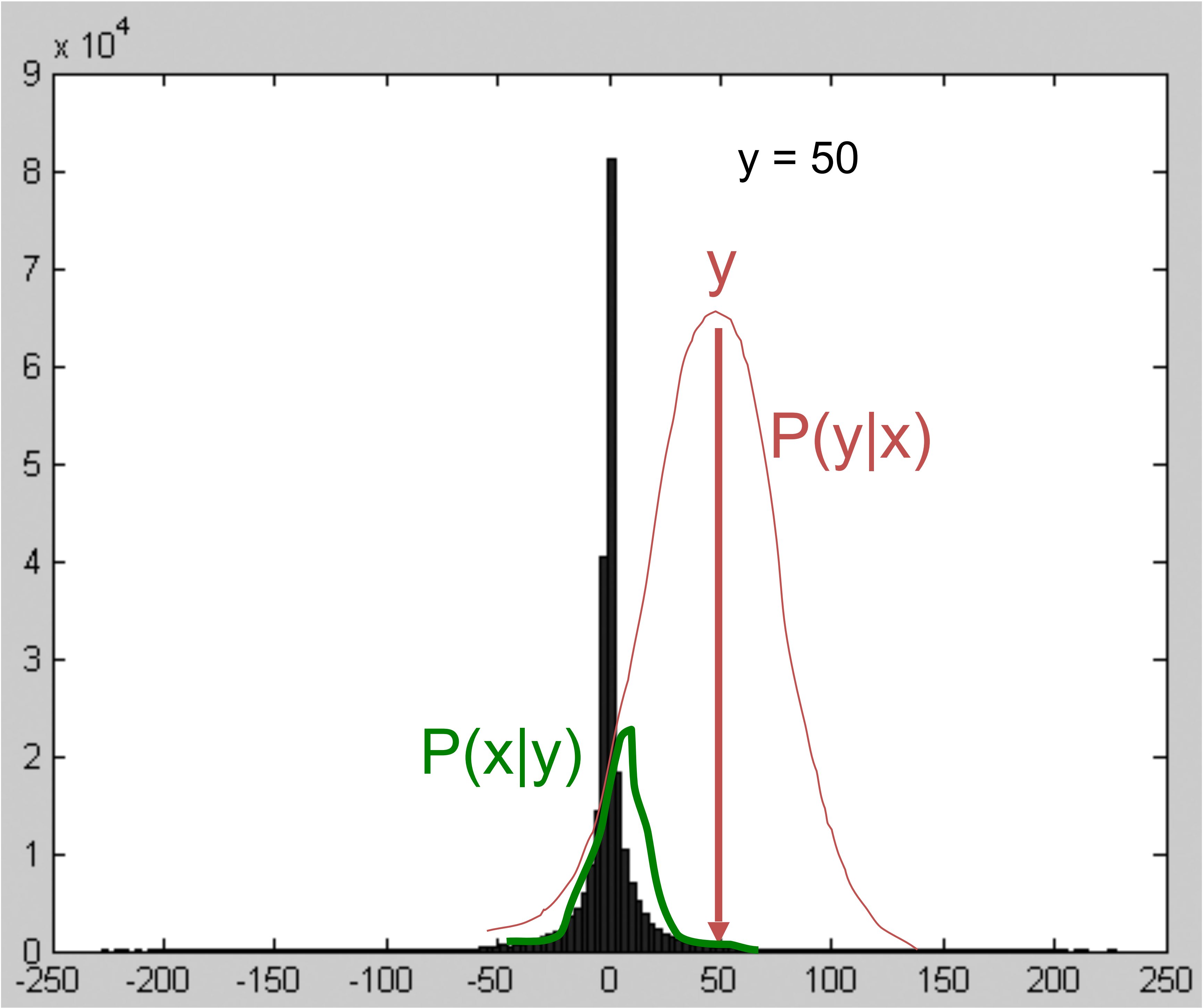


Denoising with the marginal wavelet model

Let x = bandpassed image value before adding noise.
Let y = noise-corrupted observation.

By Bayes theorem

P(x|y) ~ P(y|x) P(x)

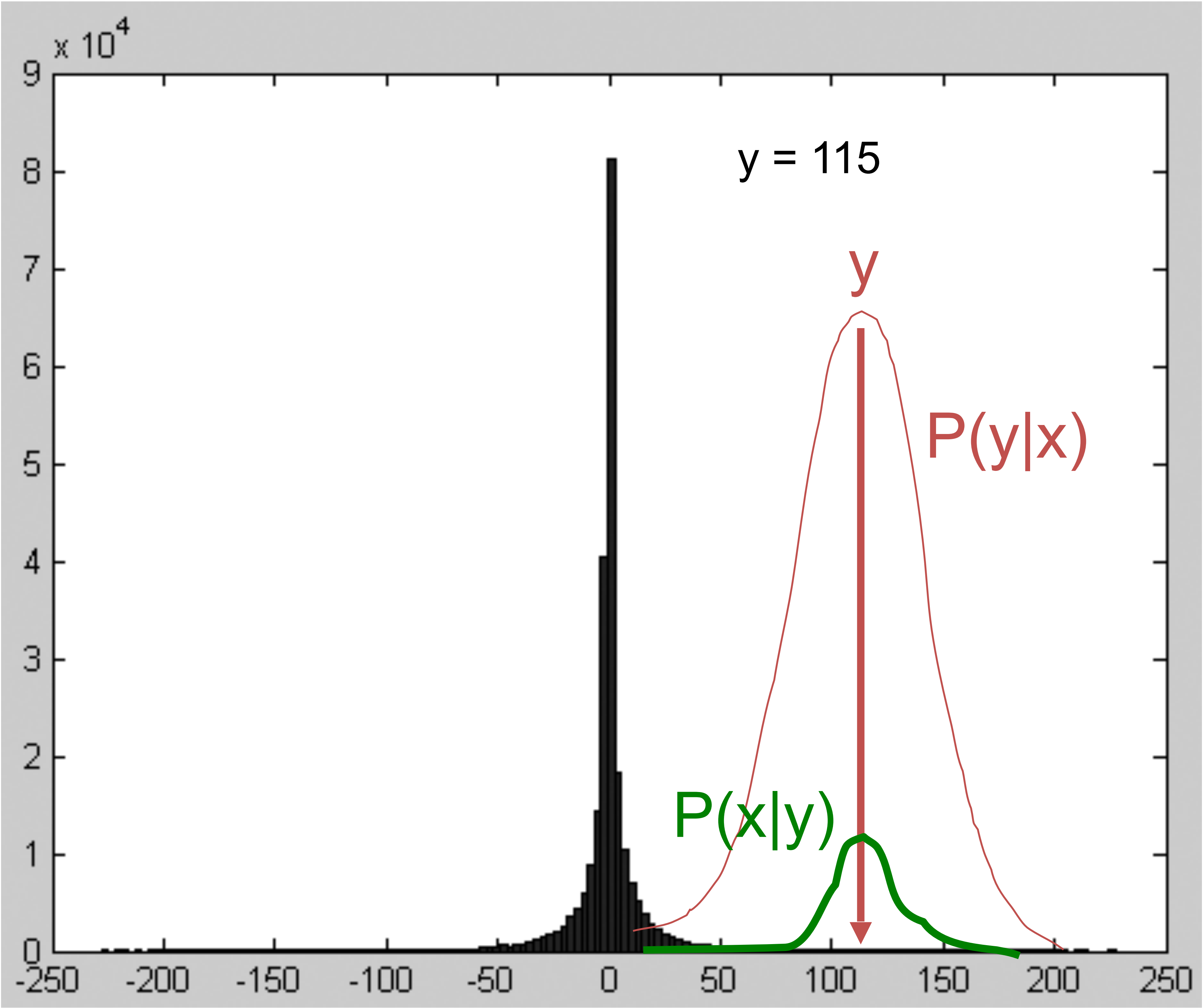


Denoising with the marginal wavelet model

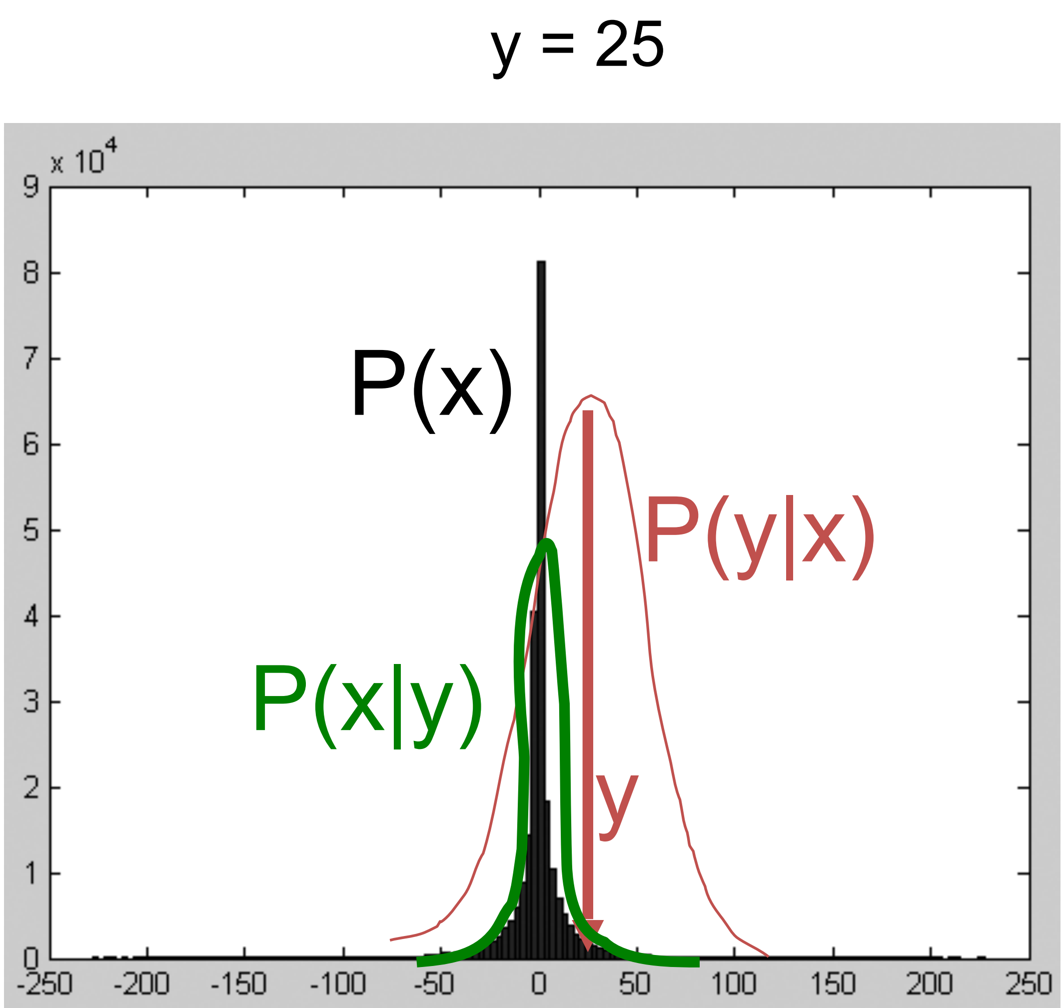
Let x = bandpassed image value before adding noise.
Let y = noise-corrupted observation.

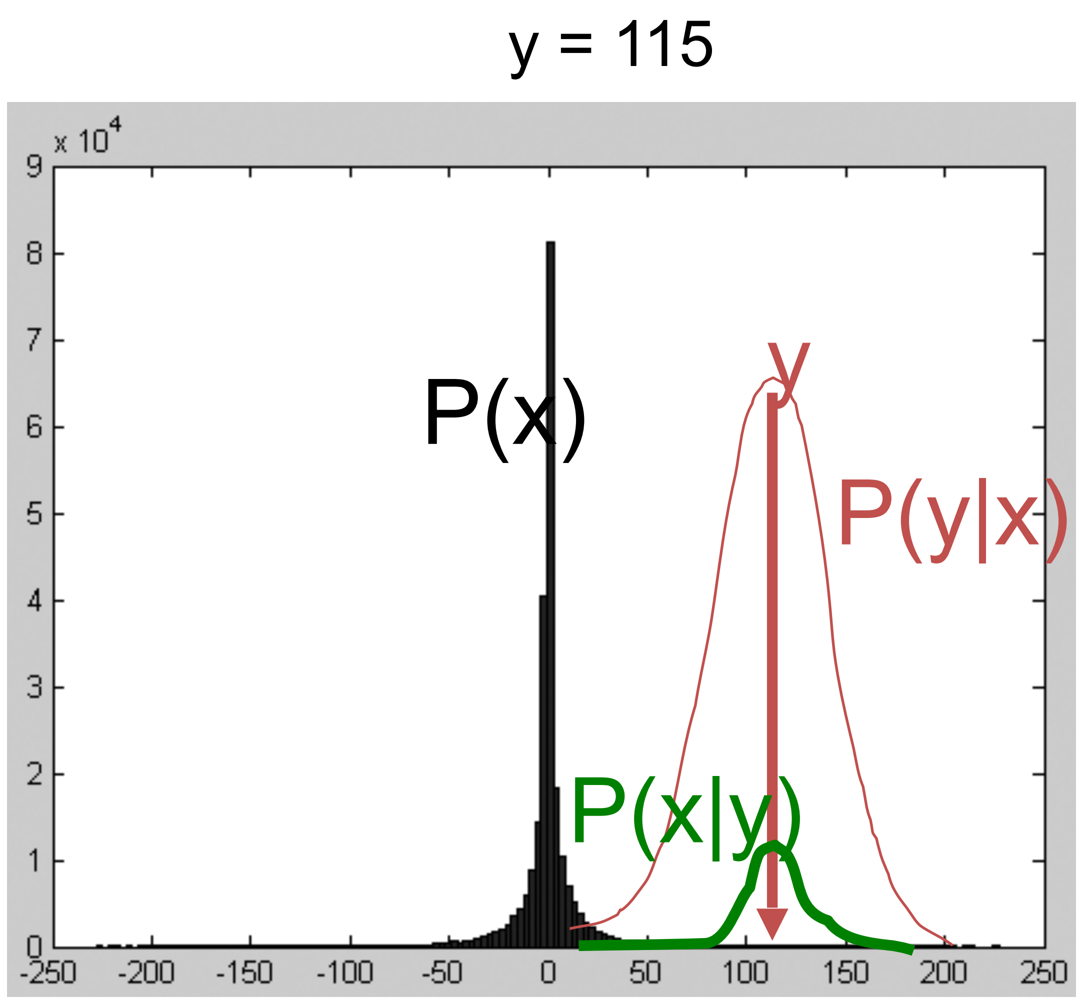
By Bayes theorem

P(x|y) ~ P(y|x) P(x)



Denoising with the marginal wavelet model





For small y: probably it is due to noise and y should be set to 0
For large y: probably it is due to an image edge and it should be kept untouched

MAP estimate, $\hat{x}$, as function of observed coefficient value, y

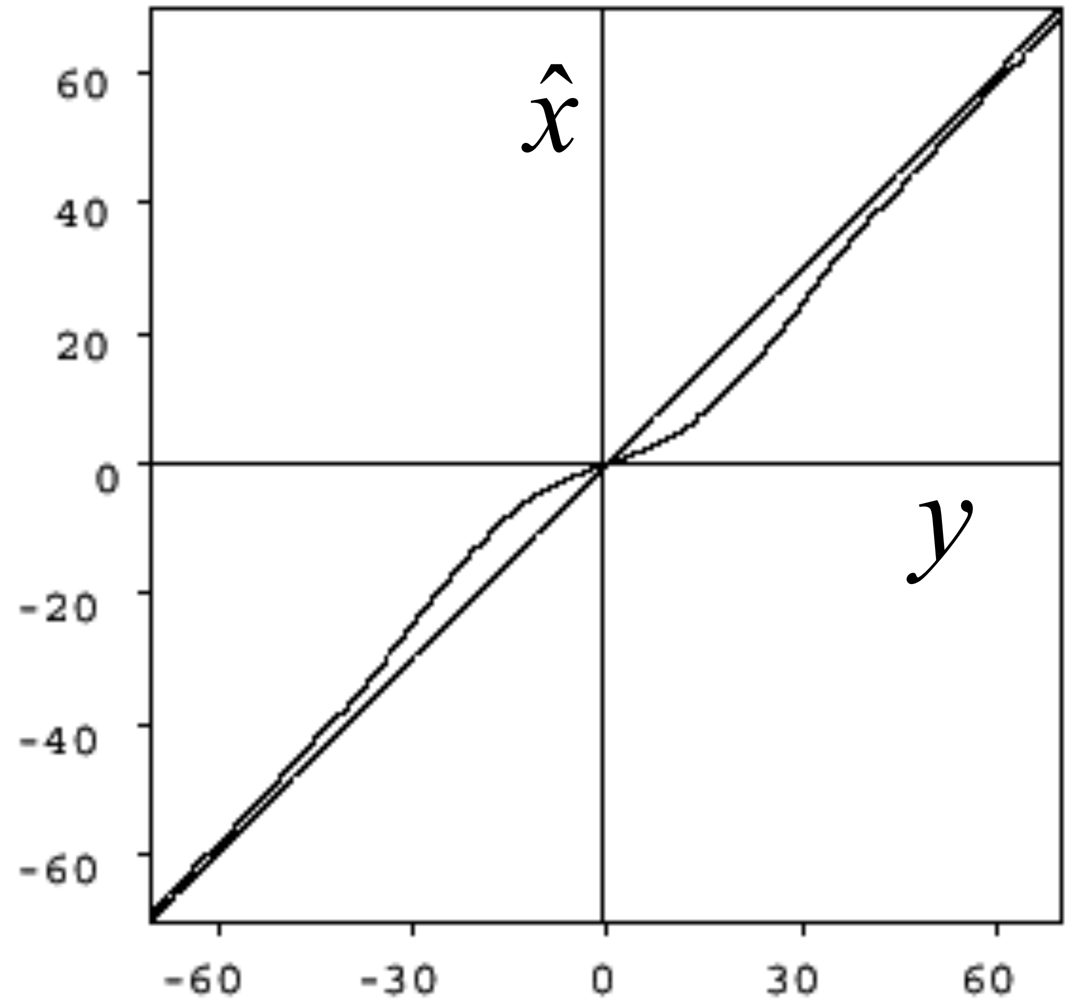


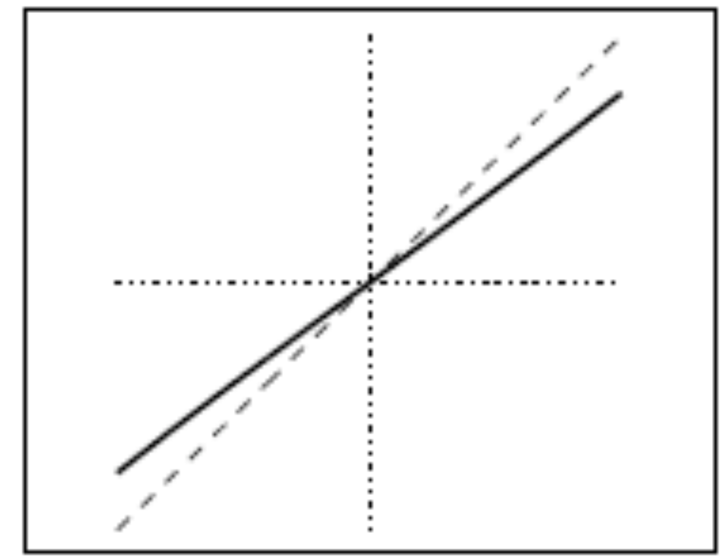
Figure 2: Bayesian estimator (symmetrized) for the signal and noise histograms shown in figure 1. Superimposed on the plot is a straight line indicating the identity function.

http://www-bcs.mit.edu/people/adelson/pub_pdfs/simoncelli_noise.pdf

Simoncelli and Adelson, Noise Removal via Bayesian Wavelet Coring

Bayesian estimate for wavelet coefficient value, for different assumed wavelet marginal distribution values of r

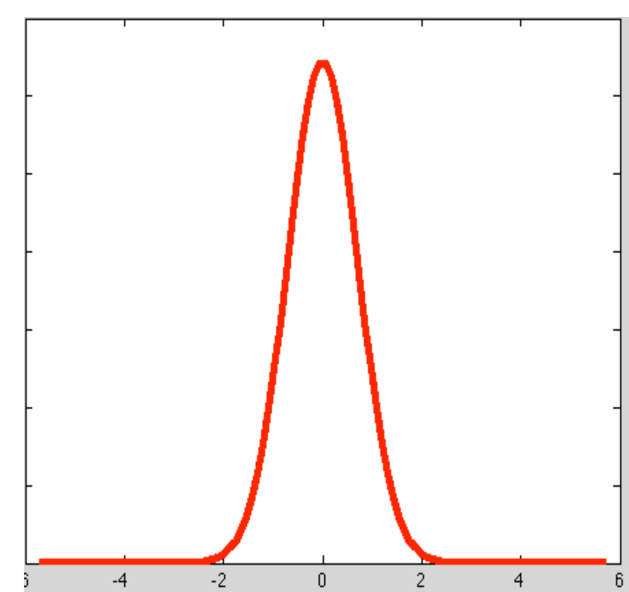
optimal denoising function
for wavelet coefficients



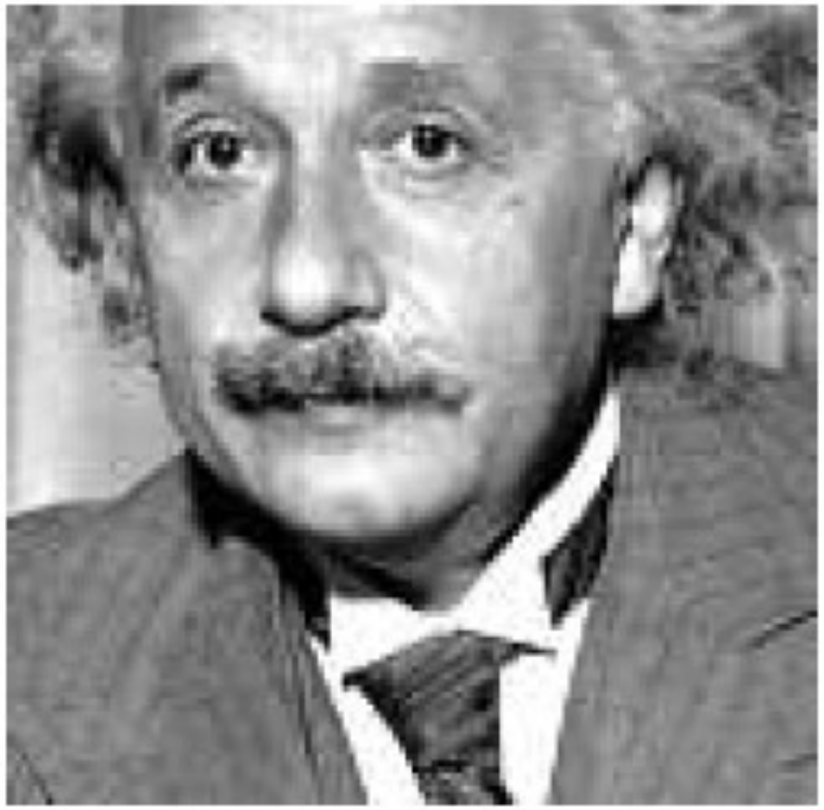
r = 2

Gaussian distribution

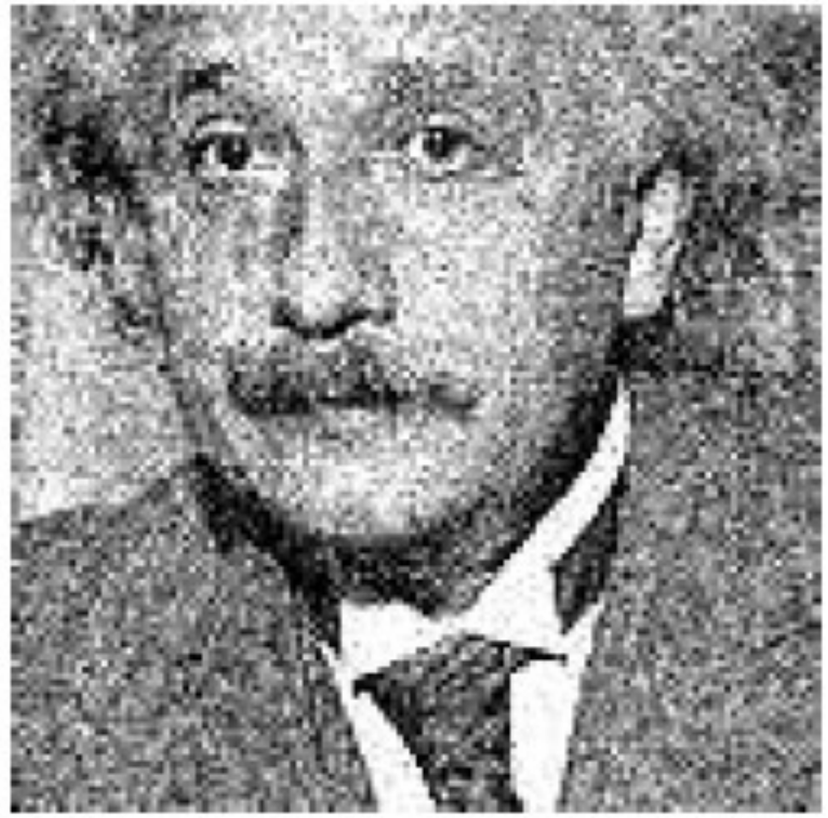
assumed wavelet
marginal distribution



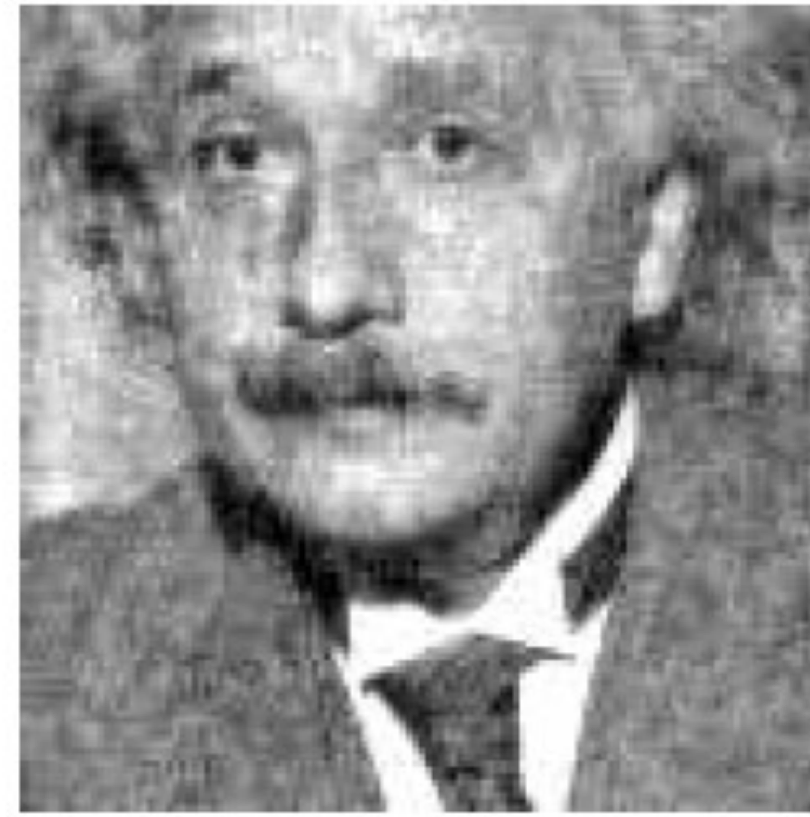
original

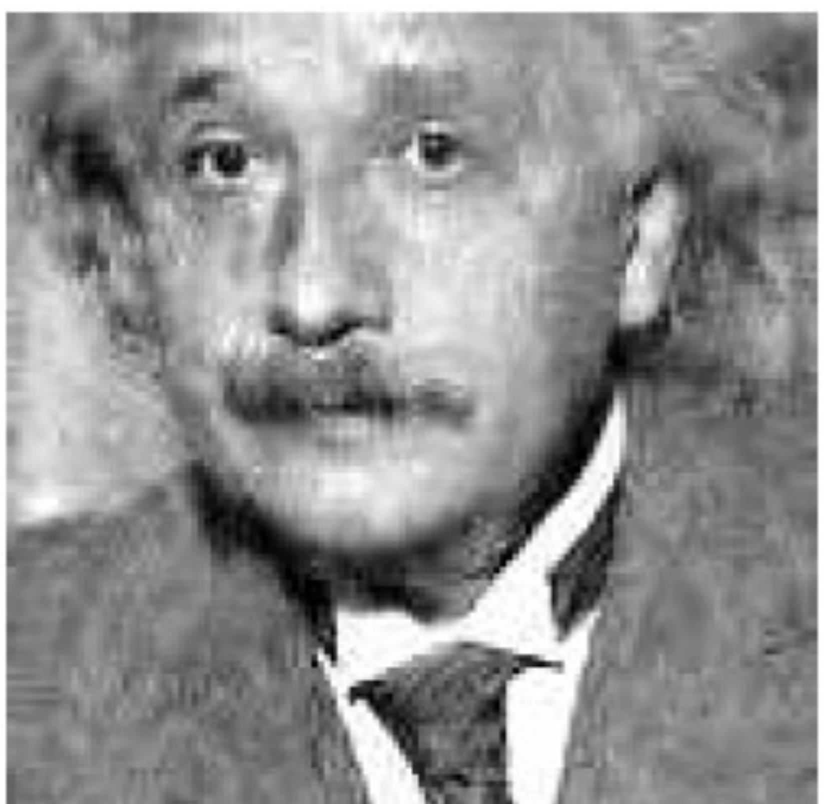


With Gaussian noise of std. dev. 21.4 added, giving PSNR=22.06



(1) Denoised with Gaussian model, PSNR=27.87





(2) Denoised with wavelet marginal model, PSNR=29.24

http://www.cns.nyu.edu/pub/eero/simoncelli05a-preprint.pdf

Statistical Image Models

- Gaussian image model
  - image synthesis
  - Wiener filter denoising
- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising
- Non-parametric model
  - image synthesis (Efros and Leung texture model)
  - Non-local means denoising

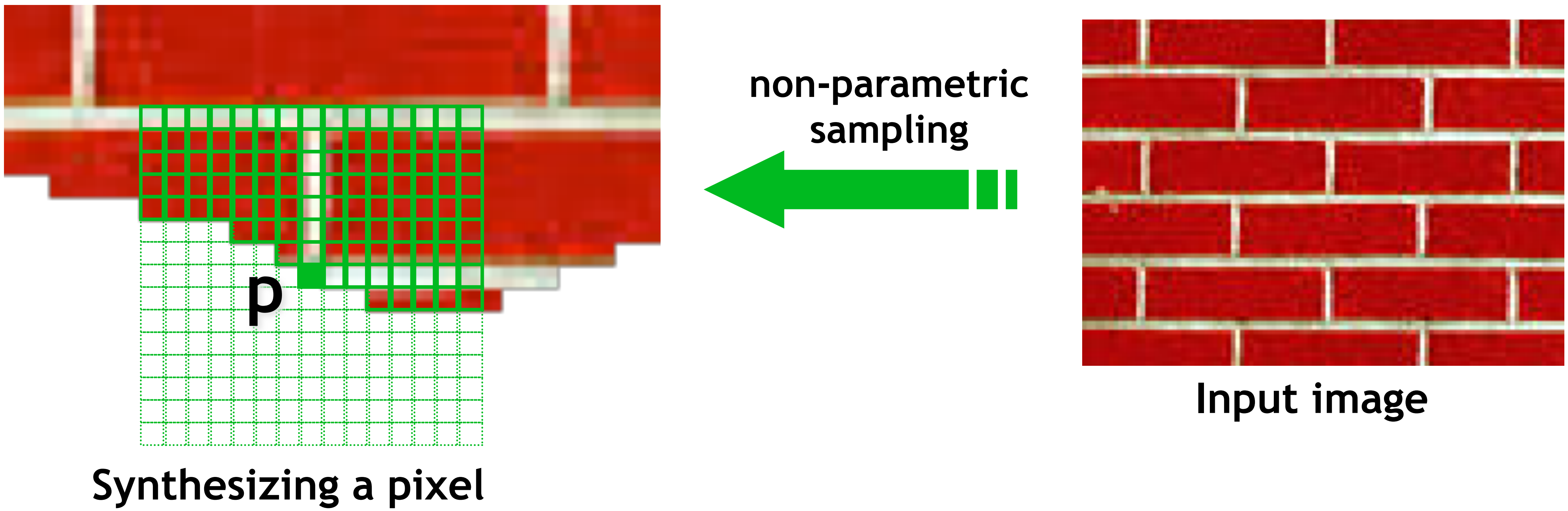
IEEE International Conference on Computer Vision, Corfu, Greece, September 1999

Texture Synthesis by Non-parametric Sampling

Alexei A. Efros and Thomas K. Leung
Computer Science Division
University of California, Berkeley
Berkeley, CA 94720-1776, U.S.A.
{efros,leungt}@cs.berkeley.edu

Image model: each image has a large set of “production rules”
If the local image values satisfy the conditions of one of the production rules, then you output a particular pixel value.

Efros & Leung Algorithm



Assuming Markov property, compute P(**p**|N(**p**))

– Building explicit probability tables is infeasible

–Instead, we *search the input image* for all similar neighborhoods — that’s our pdf for **p**

–To sample from this pdf, just pick one match at random

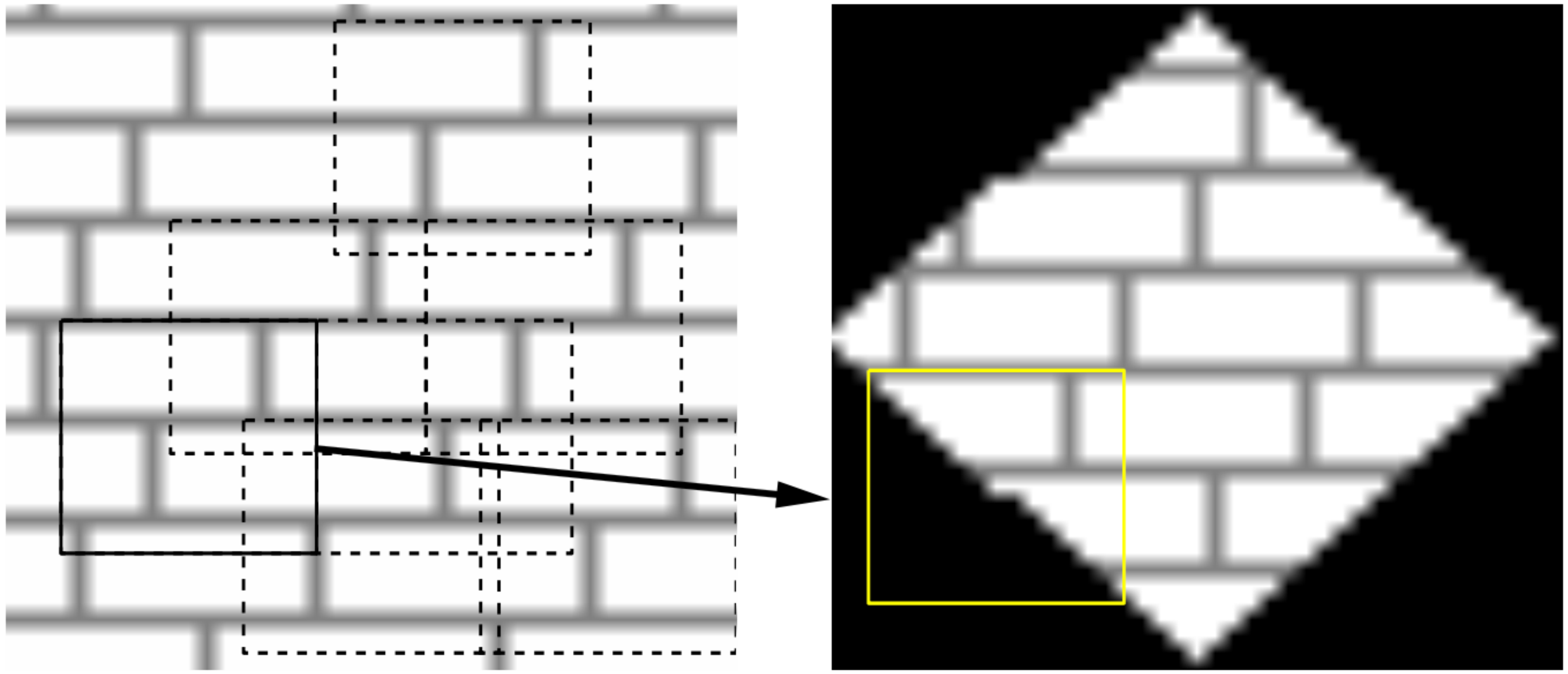
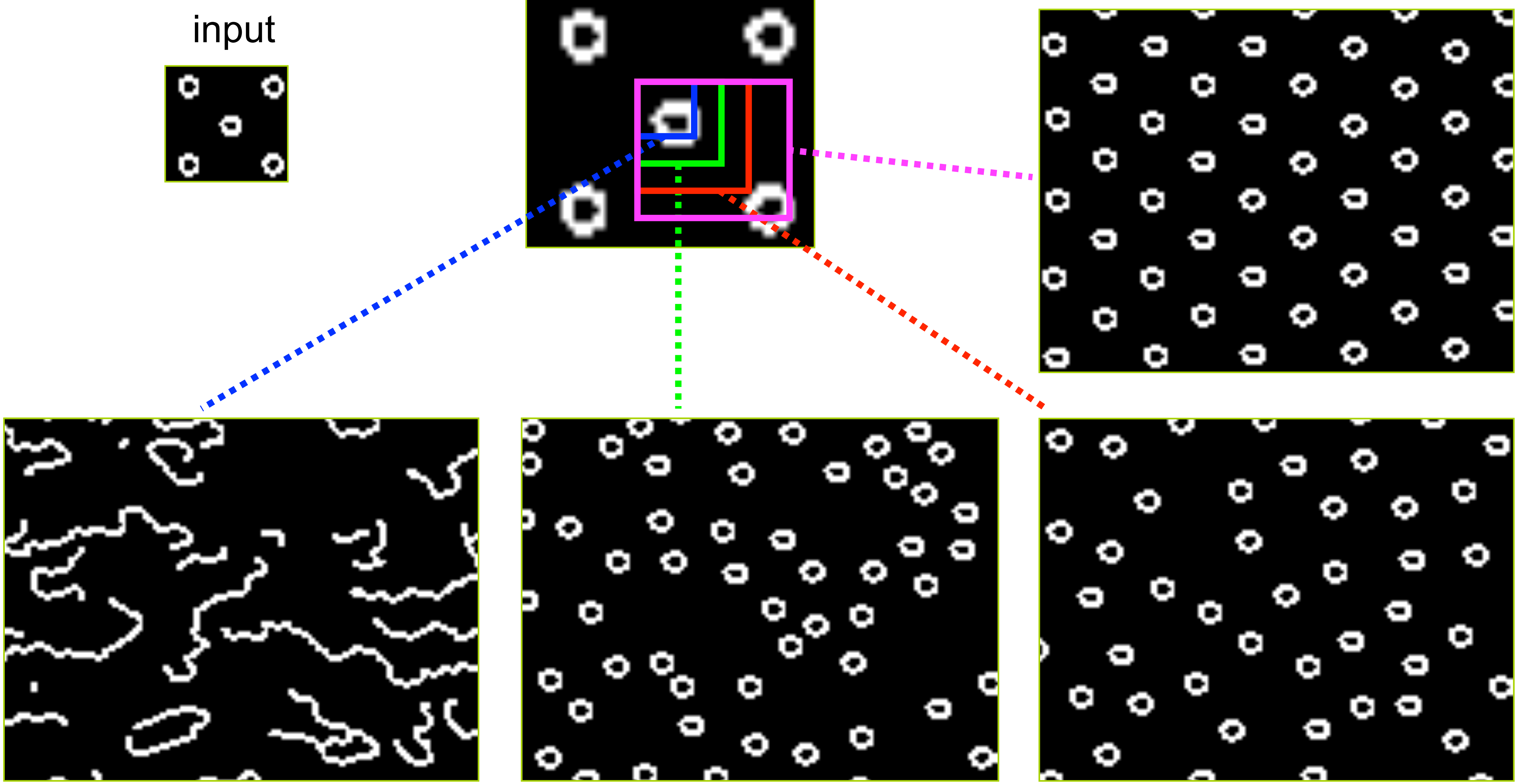
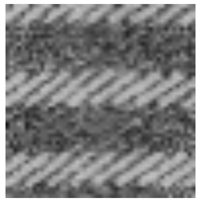


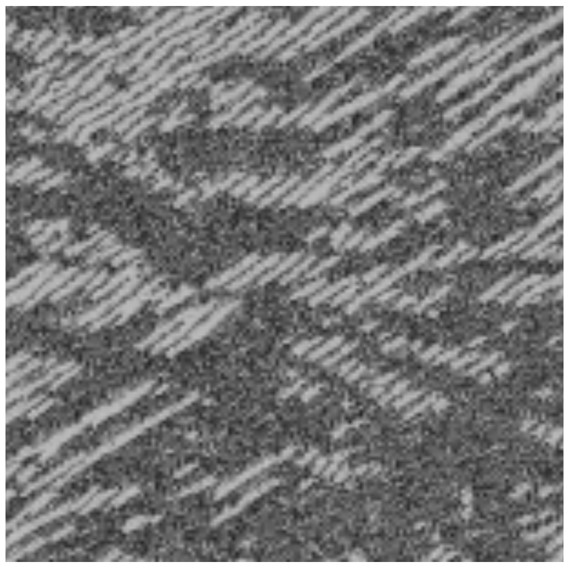
Figure 1. Algorithm Overview. Given a sample texture image (left), a new image is being synthesized one pixel at a time (right). To synthesize a pixel, the algorithm first finds all neighborhoods in the sample image (boxes on the left) that are similar to the pixel's neighborhood (box on the right) and then randomly chooses one neighborhood and takes its center to be the newly synthesized pixel.

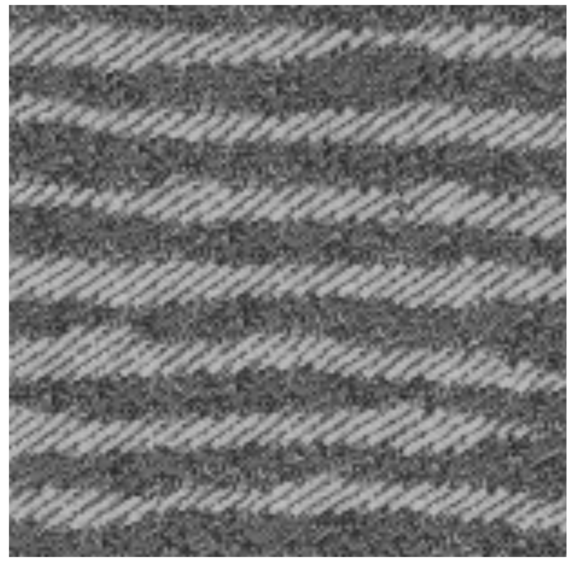
Neighborhood Window

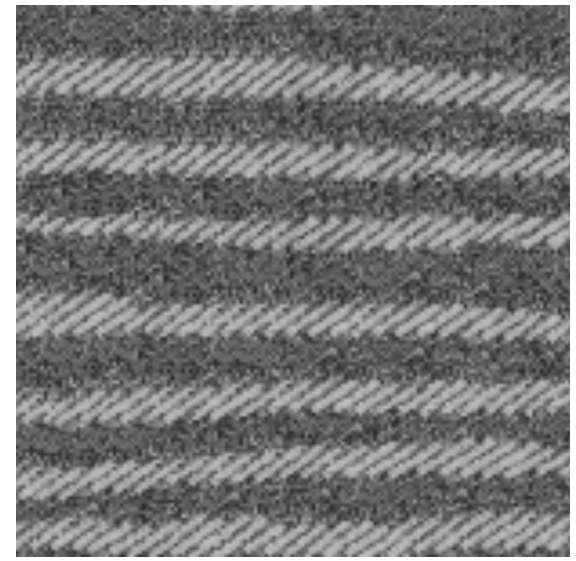


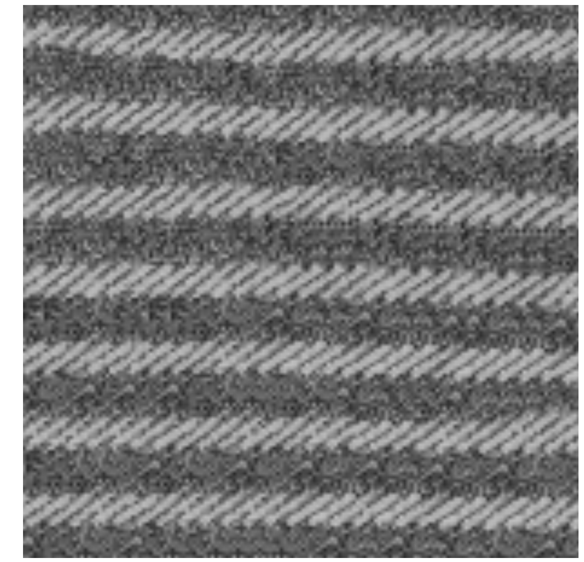
Varying Window Size

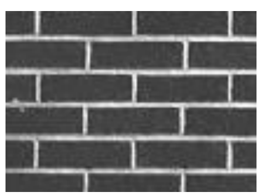


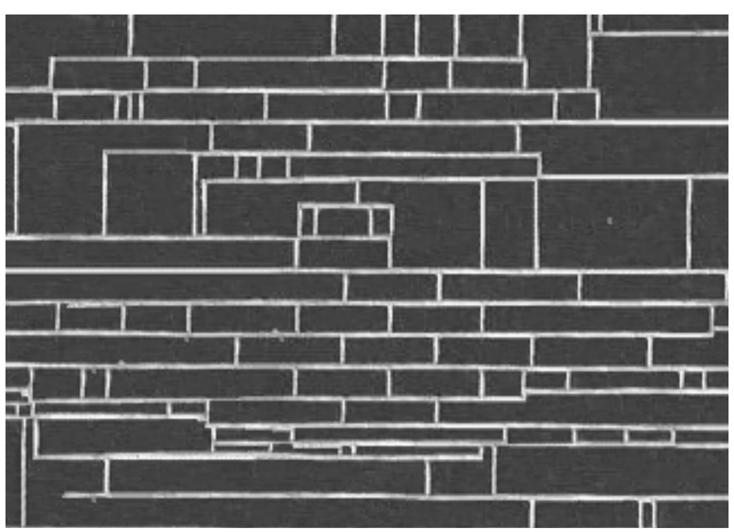


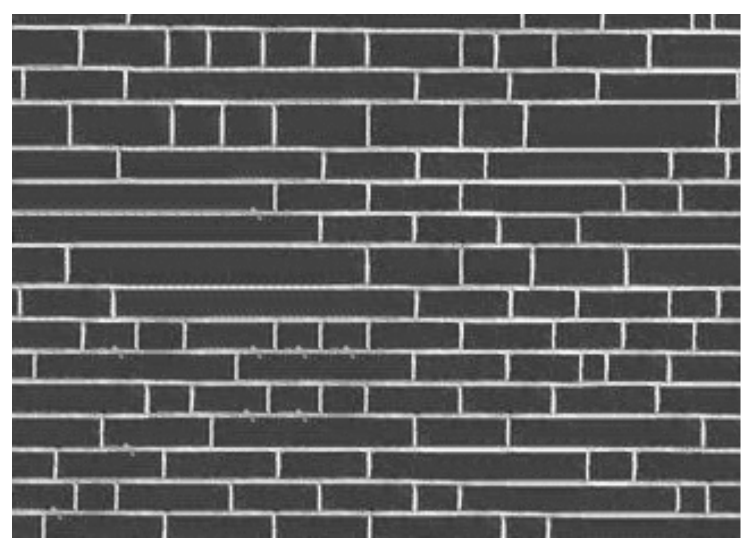


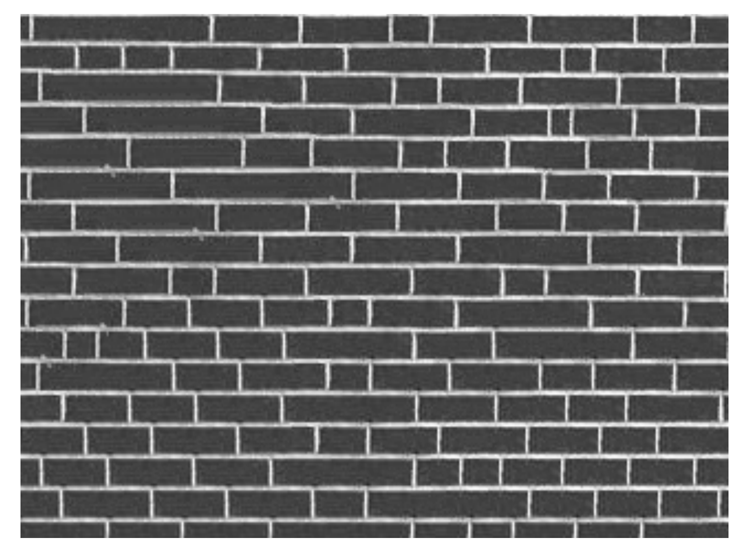








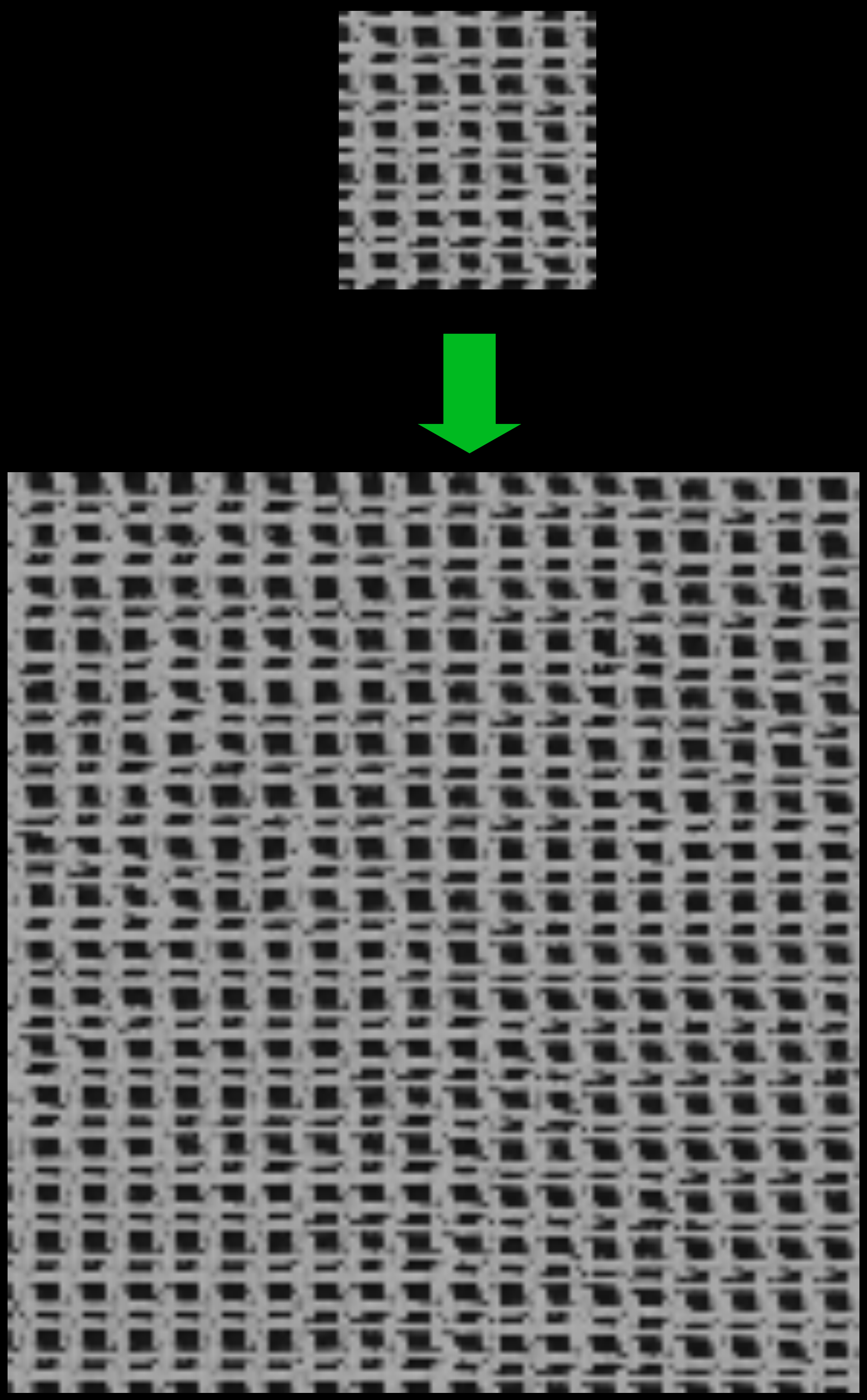




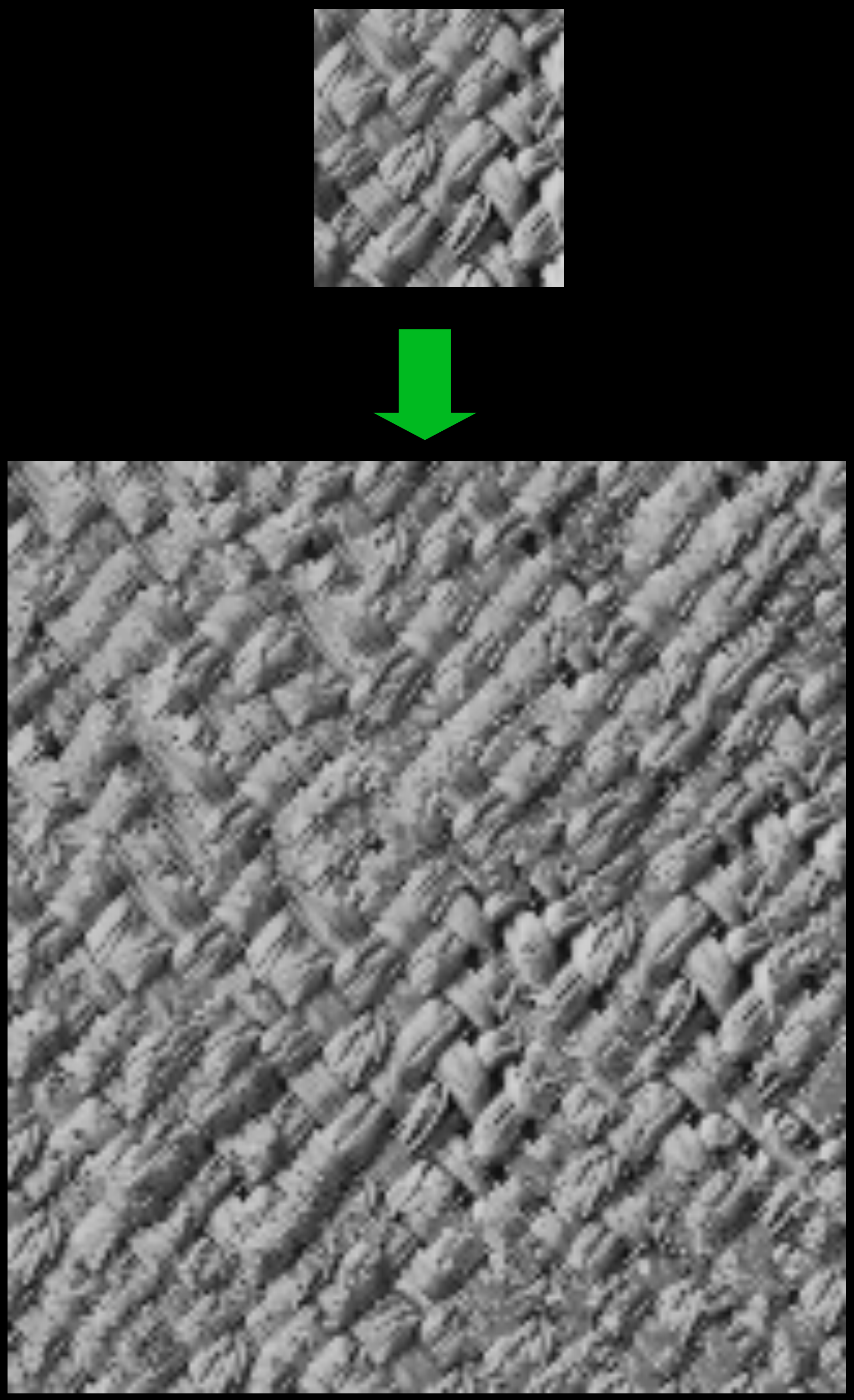
Increasing window size

Synthesis Results

french canvas

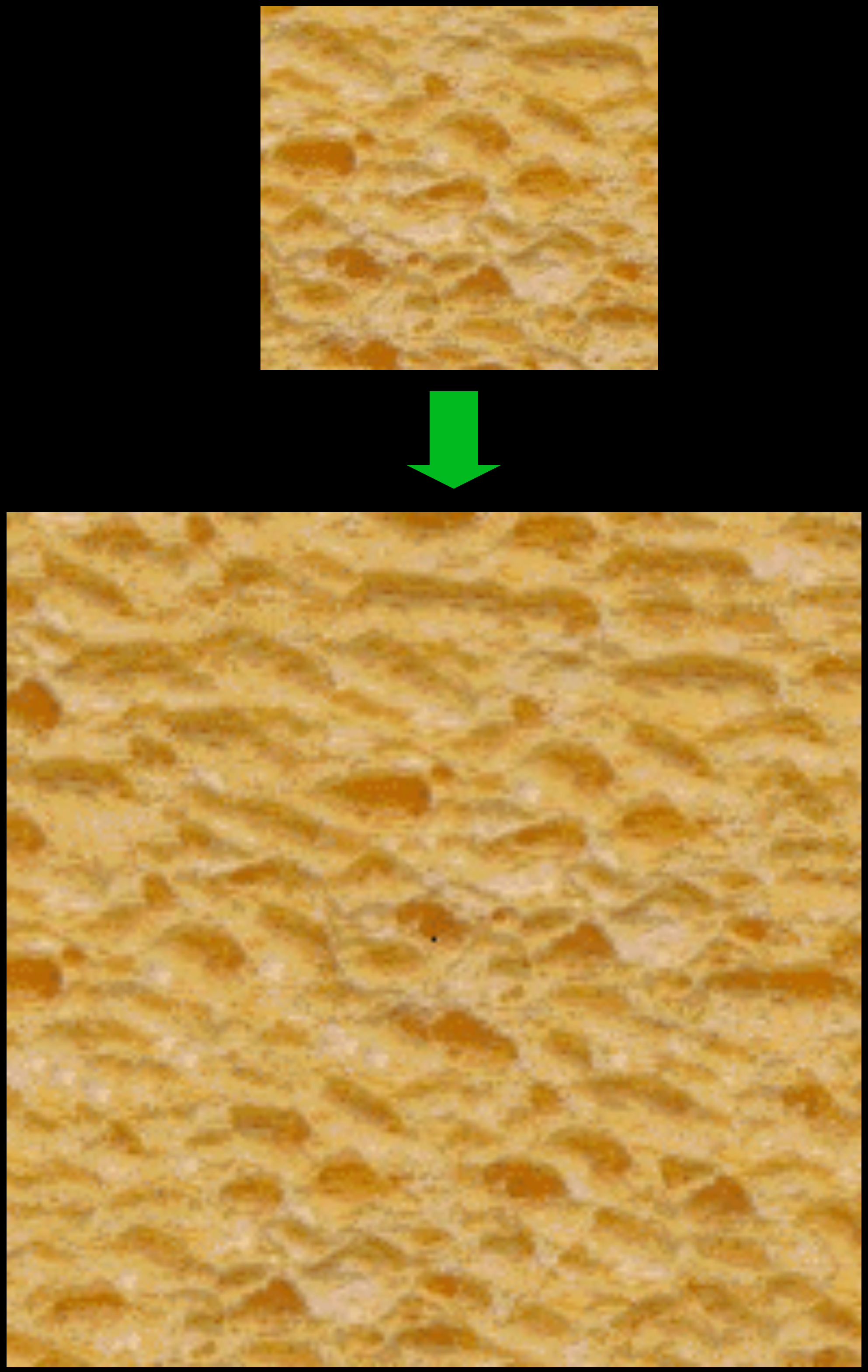


rafia weave

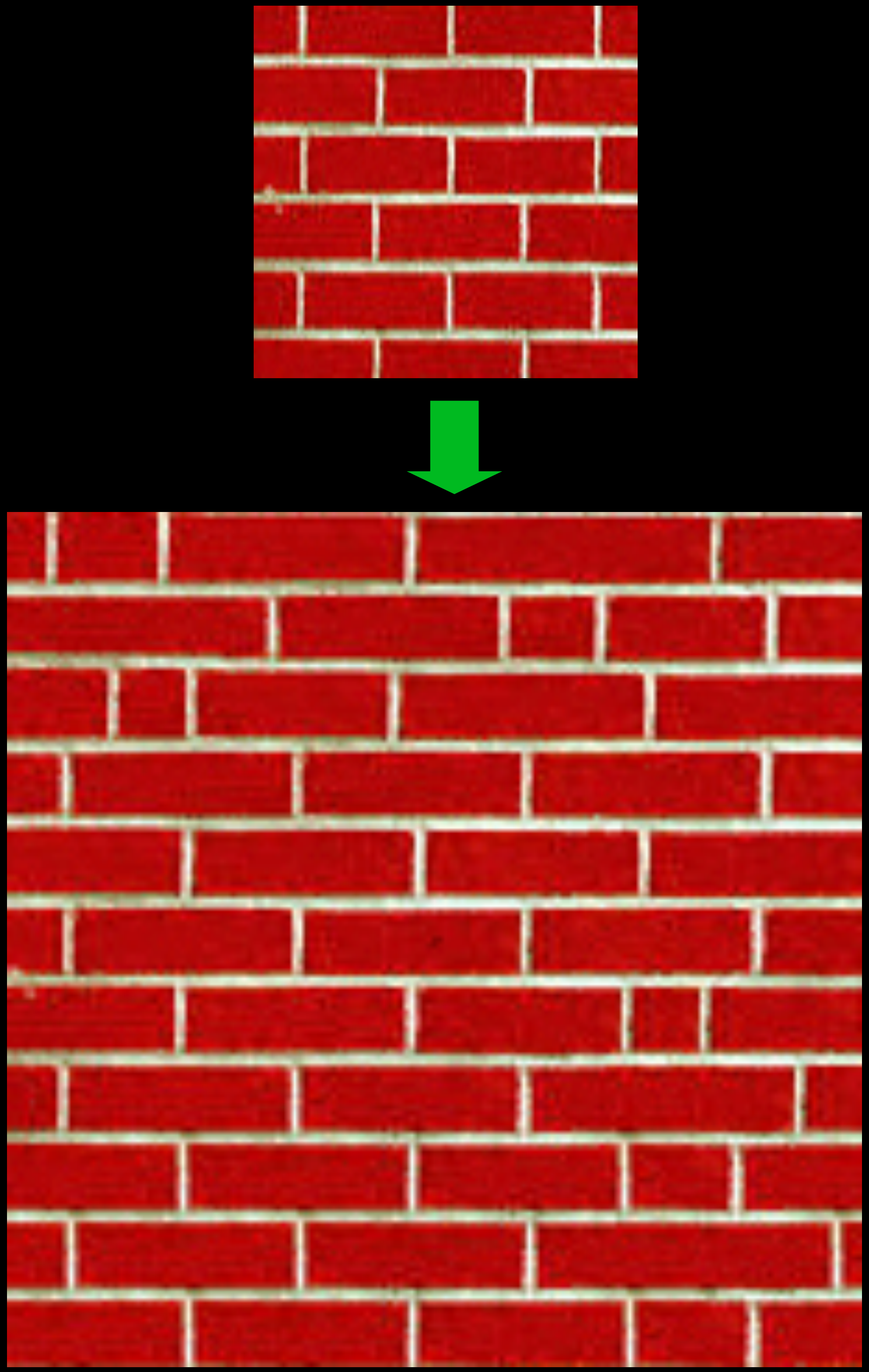


More Results

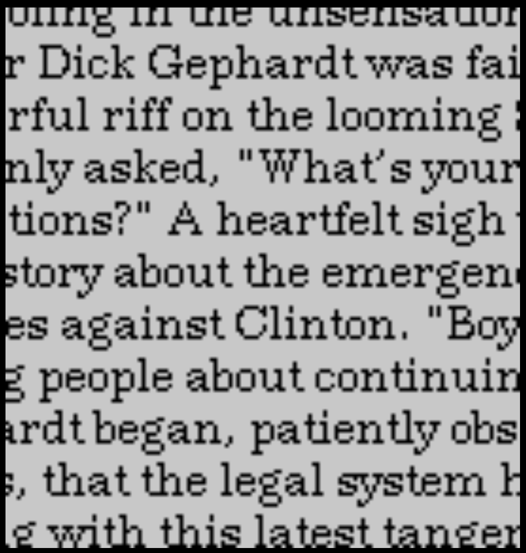
white bread

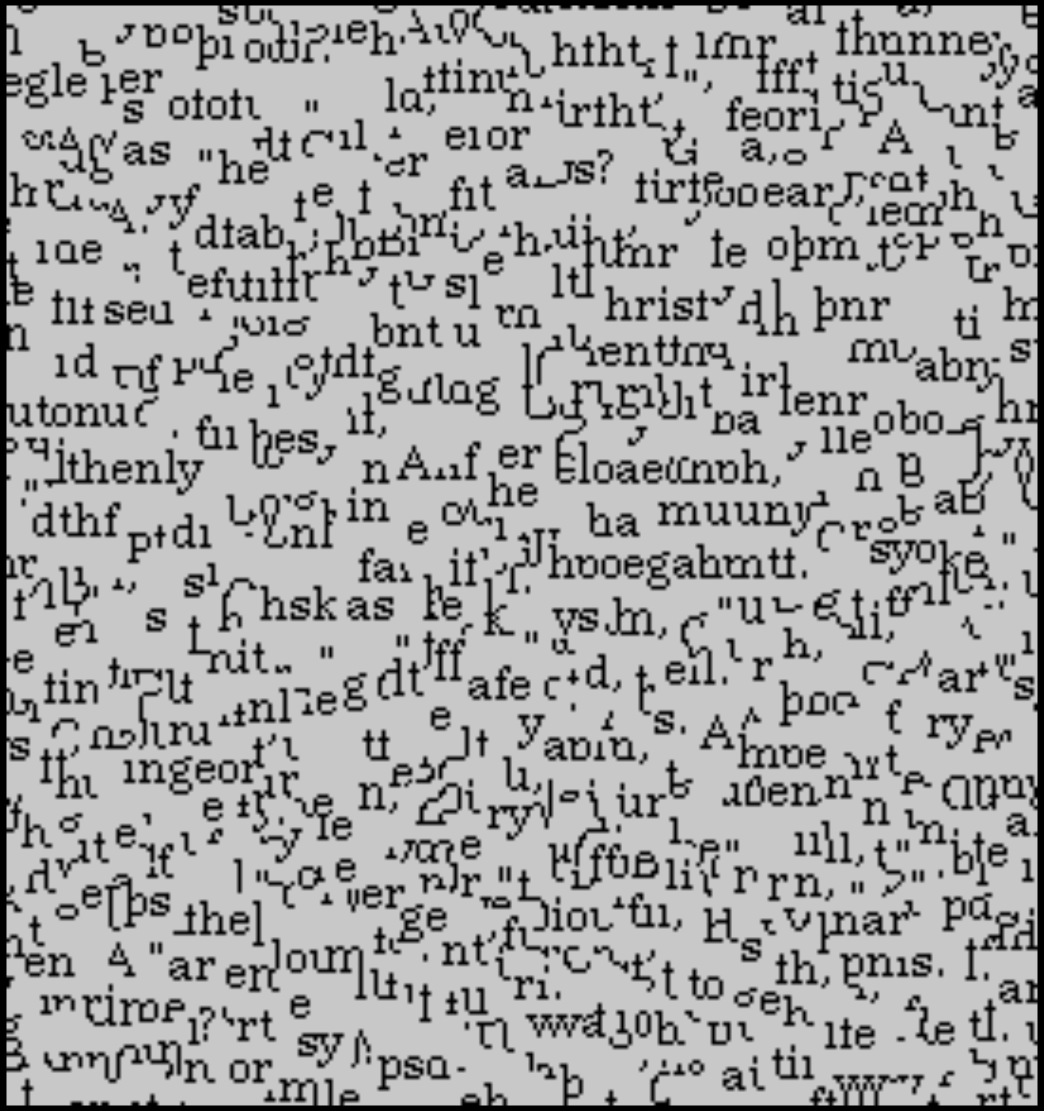


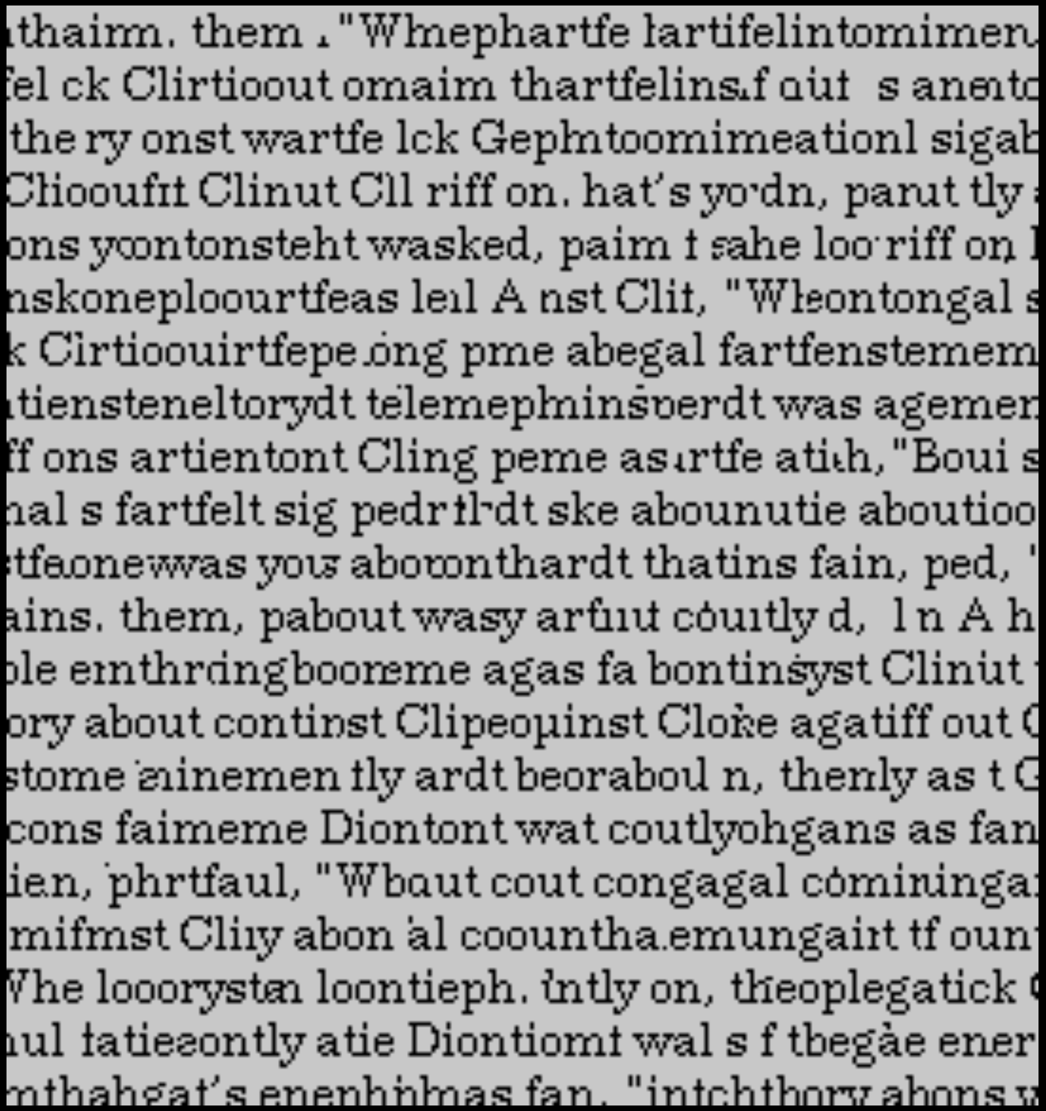
brick wall



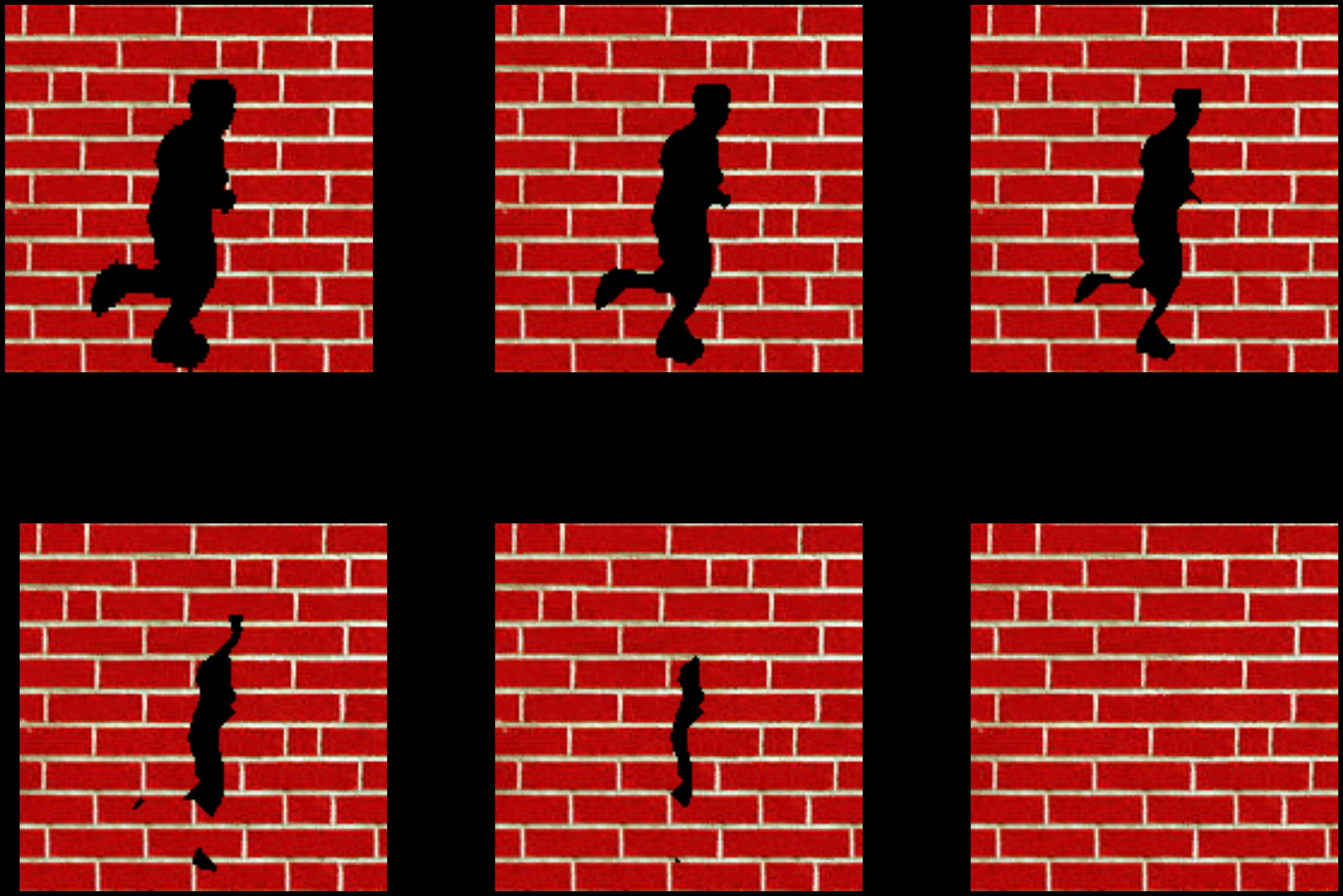
Homage to Shannon



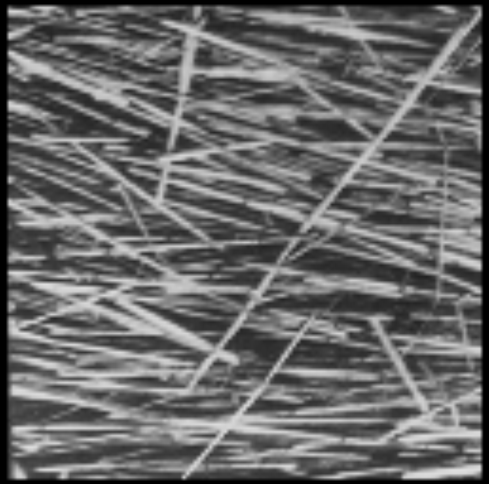


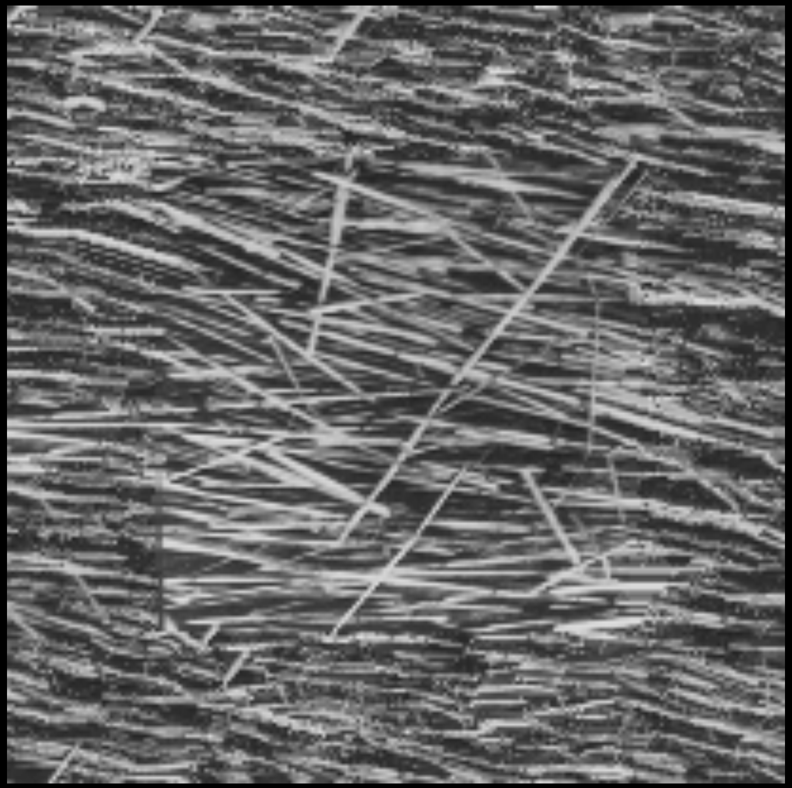


Hole Filling



Extrapolation





What denoising algorithm would result from this non-parametric model for image generation?

A denoising algorithm which implicitly assumes this non-parametric model for image generation

A non-local algorithm for image denoising

Antoni Buades, Bartomeu Coll
Dpt. Matemàtiques i Informàtica, UIB
Ctra. Valldemossa Km. 7.5,
07122 Palma de Mallorca, Spain
vdmiabc4@uib.es, tomeu.coll@uib.es

Jean-Michel Morel
CMLA, ENS Cachan
61, Av du Président Wilson
94235 Cachan, France
morel@cmla.ens-cachan.fr

Non-local means

tificial shocks which can be justified by the computation of its method noise, see [3].

3. NL-means algorithm

Given a discrete noisy image $v = \{v(i) \mid i \in I\}$, the estimated value $NL[v](i)$, for a pixel i, is computed as a weighted average of all the pixels in the image,

$$NL[v](i) = \sum_{j \in I} w(i,j) v(j),$$

$$w(i,j) = \frac{1}{Z(i)} e^{-\frac{\|v(\mathcal{N}_i) - v(\mathcal{N}_j)\|_{2,a}^2}{h^2}},$$

where $Z(i)$ is the normalizing constant

$$Z(i) = \sum_j e^{-\frac{\|v(\mathcal{N}_i) - v(\mathcal{N}_j)\|_{2,a}^2}{h^2}}$$

and the parameter h acts as a degree of filtering. It controls the decay of the exponential function and therefore the decay of the weights as a function of the Euclidean distances.

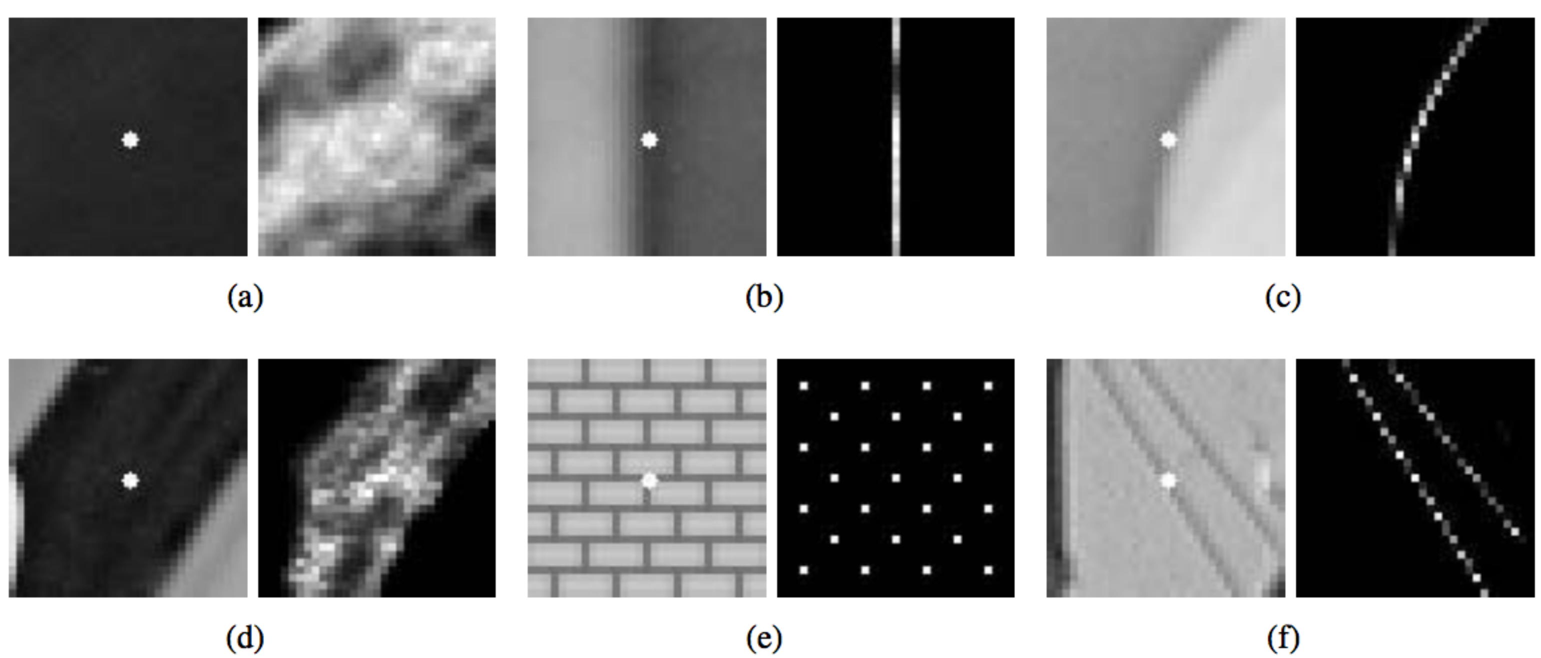


Figure 2. Display of the NL-means weight distribution used to estimate the central pixel of every image. The weights go from 1(white) to zero(black).

Non-local means

Noisy original

Wiener filter output

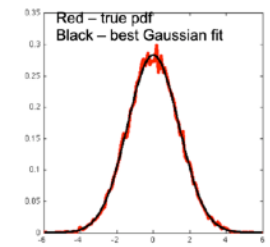
Total variation output

Non-local means output

gure 5. Denoising experience on a natural image. From left to right and from top to bottor
tandard deviation 20), Gauss filtering, anisotropic filtering, Total variation, Neighborhood f
eans algorithm. The removed details must be compared with the method noise experience, F

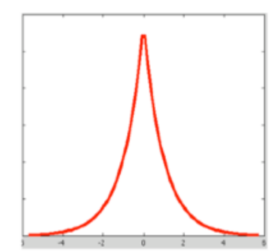
Statistical Image Models

- Gaussian image model
  - image synthesis
  - Wiener filter denoising



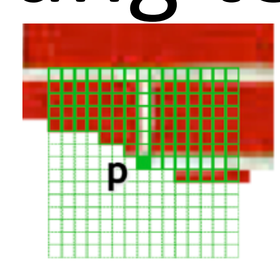
subband statistics (or statistics of any linear combination of pixels)

- Kurtotic wavelets model
  - image synthesis
  - Bayesian denoising



subband statistics

- Non-parametric model
  - image synthesis (Efros and Leung texture model)
  - Non-local means denoising



production rules