Lecture 5

Spatial Linear Filters

Bill Freeman, Phillip Isola

Spring 2022

Today's lecture: Spatial Filtering

Problem set 1 due today. Problem set 2 out today (due Thursday of next week).

This week: Tuesday: spatial filtering. Thursday: temporal filtering

Next week: no class Tuesday, then on Thursday: image pyramids.

Today:

- Fourier transforms and signal processing, continued
- Fourier processing by human visual system
- Spatial digital filters

The Discrete Fourier transform

2D Discrete Fourier Transform (DFT) transforms an image $f\,[n,m]$ into $F\,[u,v]$ as:

$$\boxed{F[u,v]} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \boxed{f[n,m]}\ \boxed{\exp\left(-2\pi j\left(\frac{un}{N}+\frac{vm}{M}\right)\right)}$$

The inverse Discrete Fourier transform

2D Discrete Fourier Transform (DFT) transforms an image *f* [*n*,*m*] into *F* [*u*,*v*] as:

$$F[u,v] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m] \exp\left(-2\pi j \left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

The inverse of the 2D DFT is:

$$f[n,m] = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} F[u,v] \exp\left(+2\pi j \left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

Visualizing the image Fourier transform

$$\boxed{F[u,v]} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \boxed{f[n,m]}\ \boxed{\exp\left(-2\pi j\left(\frac{un}{N}+\frac{vm}{M}\right)\right)}$$

The values of $F\,[u,v]$ are complex.

Using the real and imaginary components:

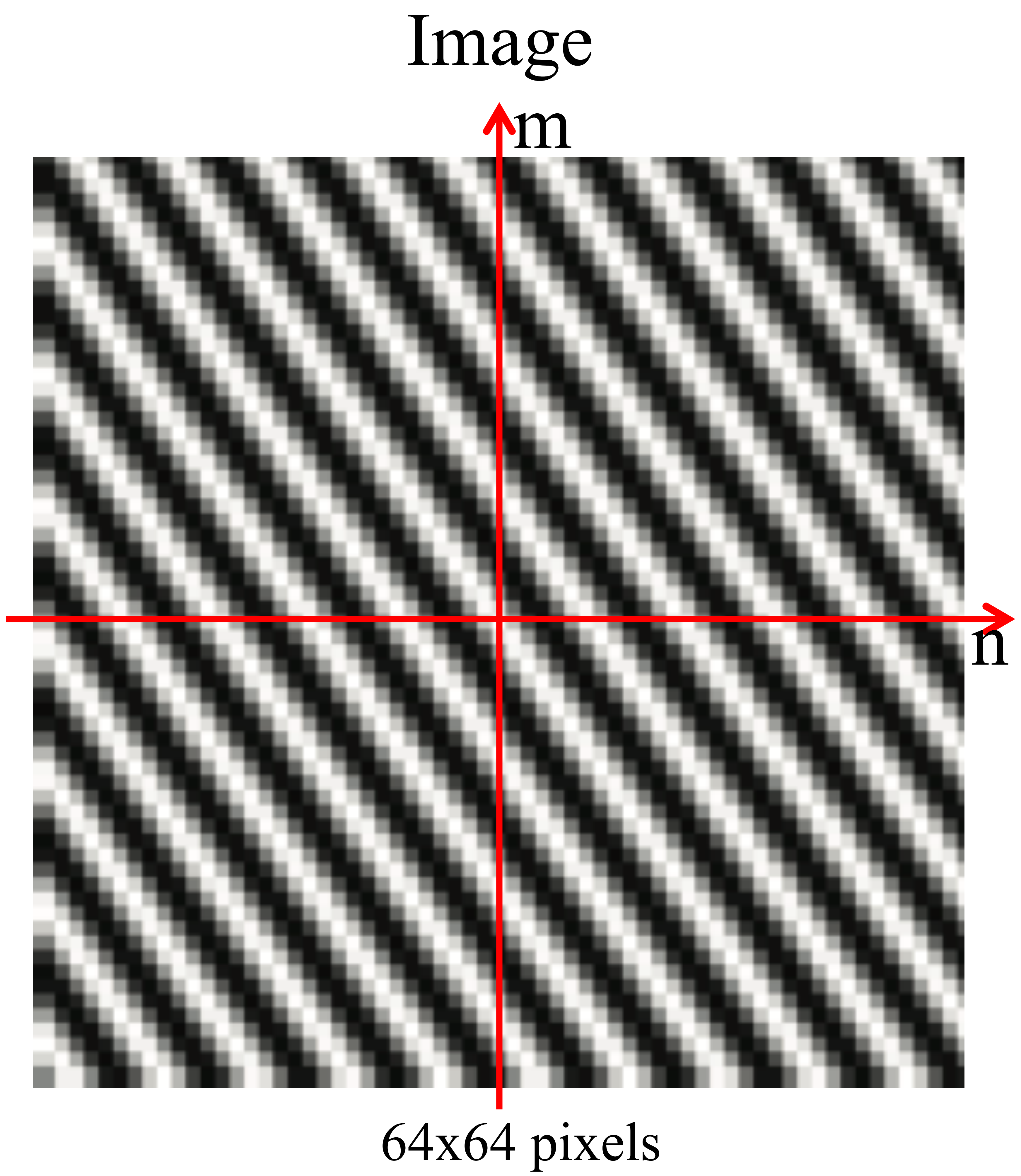
$$F\,[u,v] = Re\,\{F\,[u,v]\} + j\,Imag\,\{F\,[u,v]\}$$

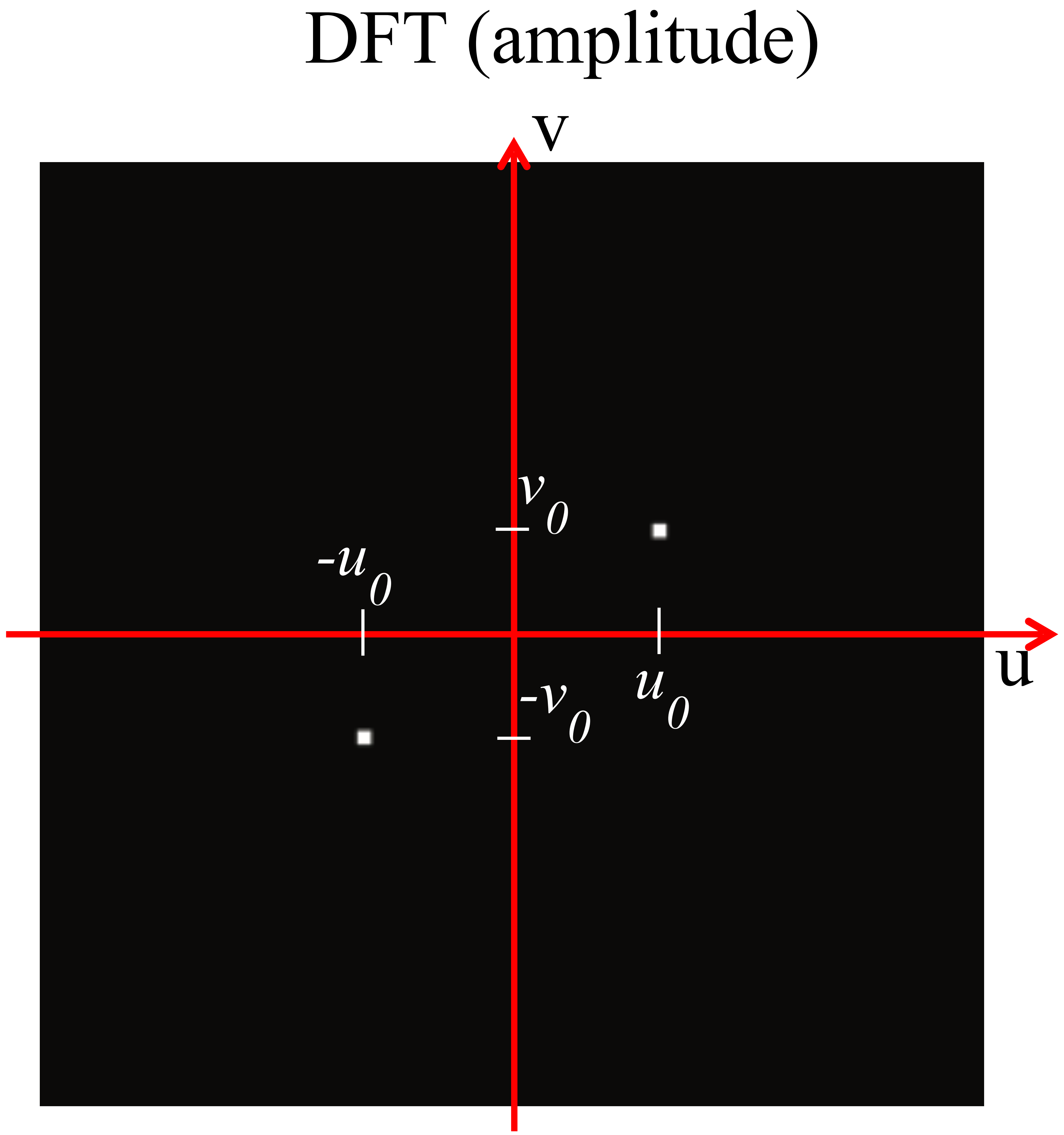
Or using a polar decomposition:

$$F\,[u,v] = A\,[u,v]\,\exp\,(j\,\theta\,[u,v])$$

Amplitude Phase

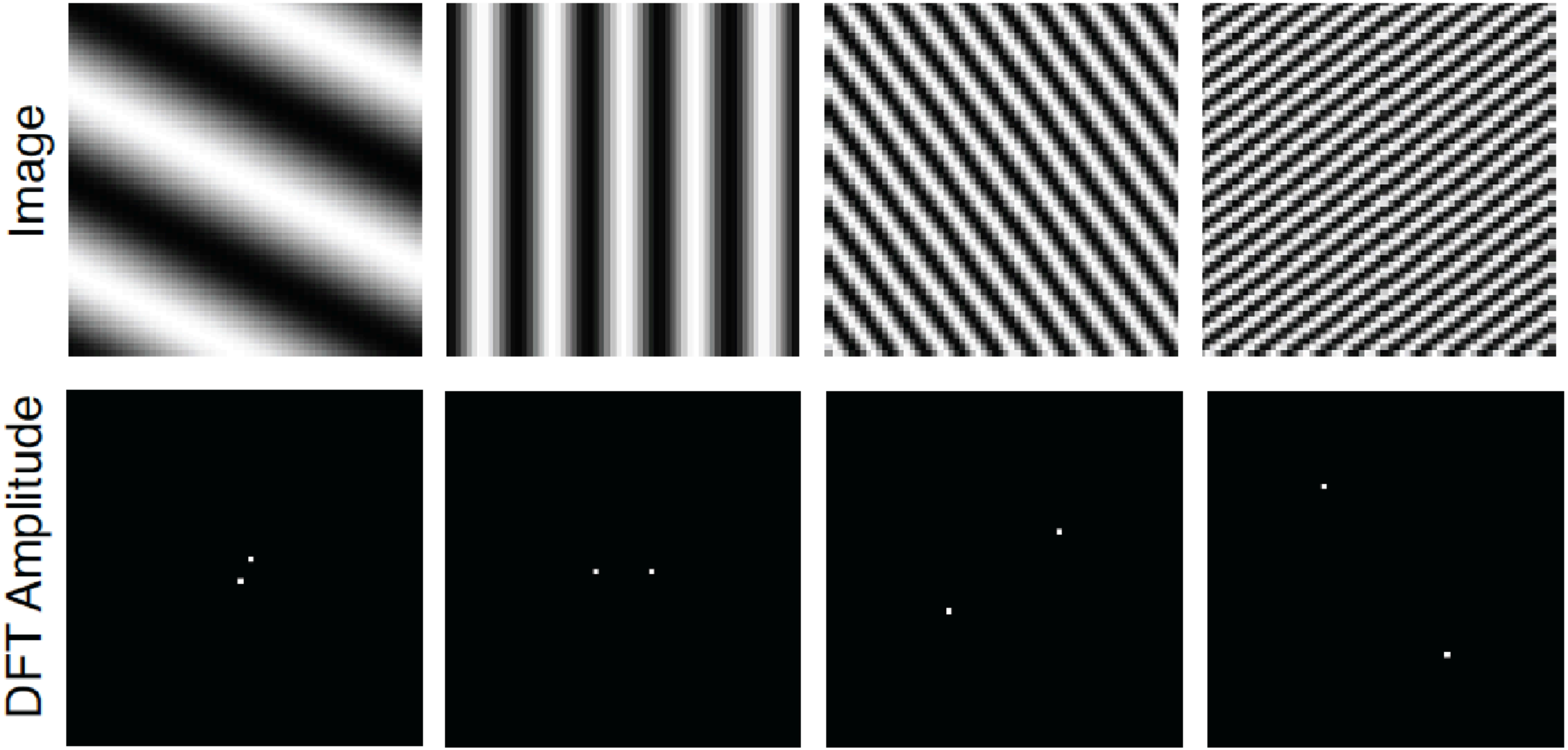
Simple Fourier transforms





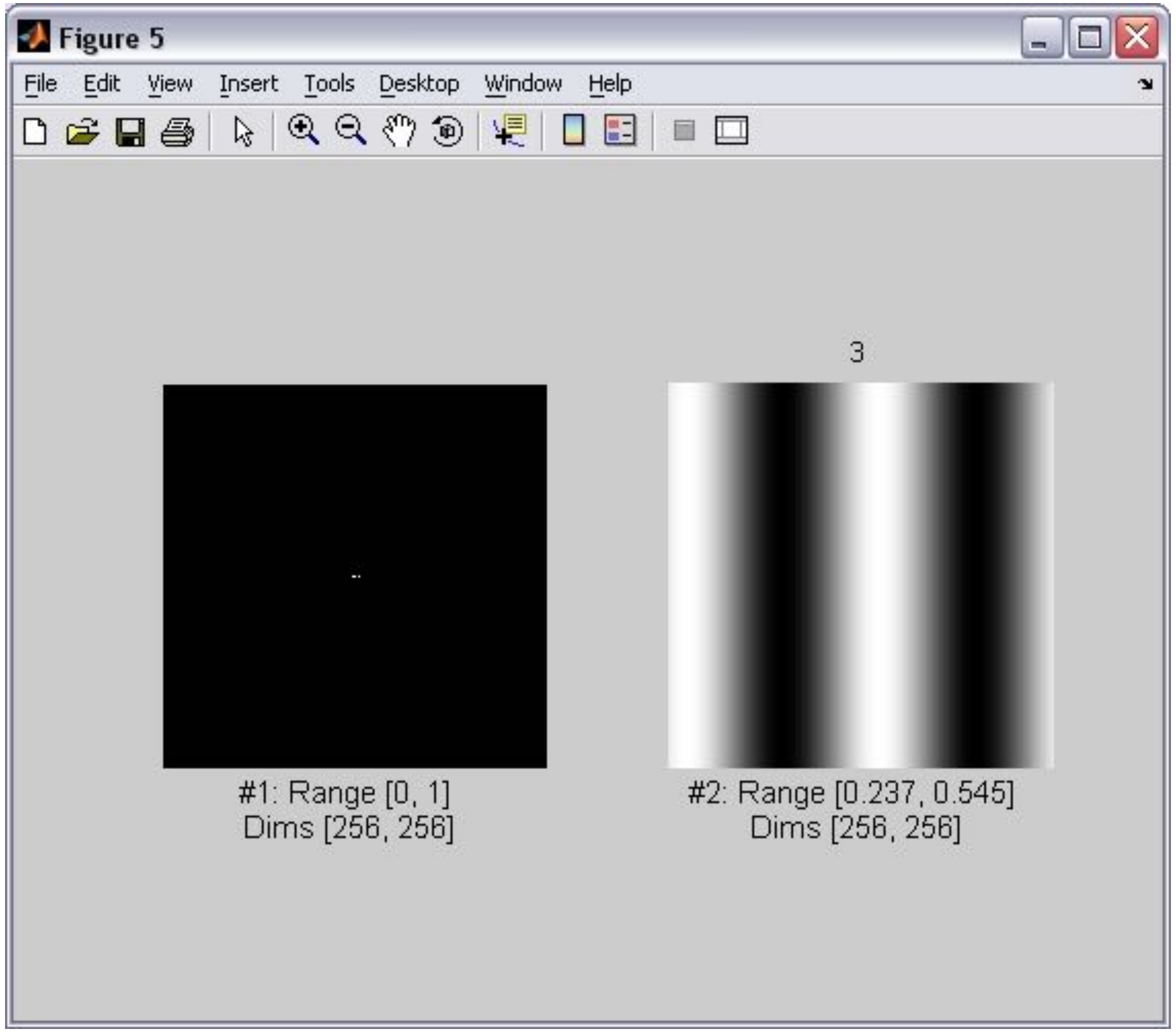
$$\cos\left(2\pi\left(\frac{u_0\, n}{N} + \frac{v_0\, m}{M}\right)\right) \longleftrightarrow \frac{1}{2}\left(\delta\left[u - u_0, v - v_0\right] + \delta\left[u + u_0, v + v_0\right]\right)$$

Simple Fourier transforms



Images are 64x64 pixels. The wave is a cosine, therefore DFT phase is zero.

3



Now, an analogous sequence of images, but selecting Fourier components in descending order of magnitude.

5

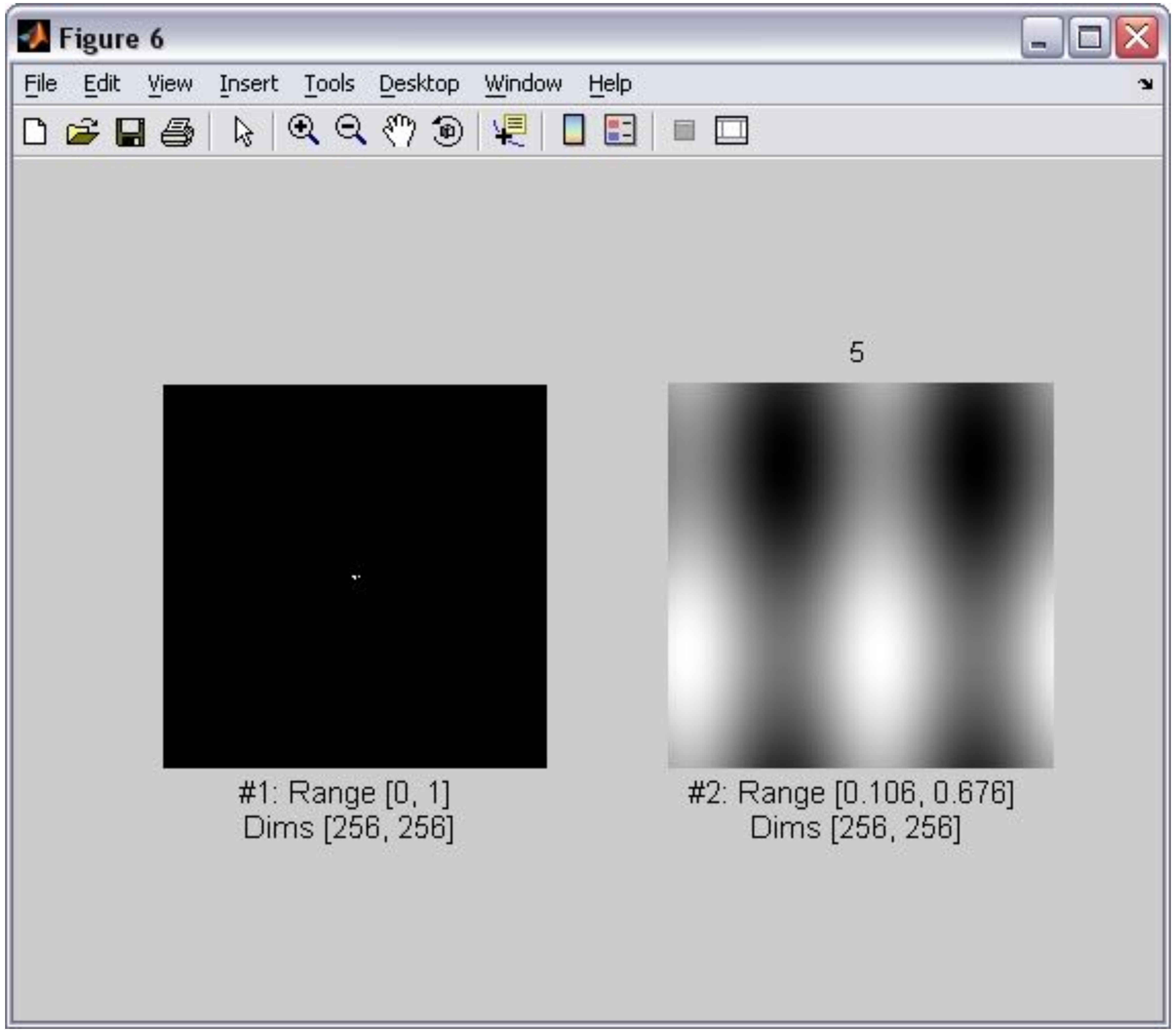


Figure 6
File Edit View Insert Tools Desktop Window Help
5
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.106, 0.676]
Dims [256, 256]

9

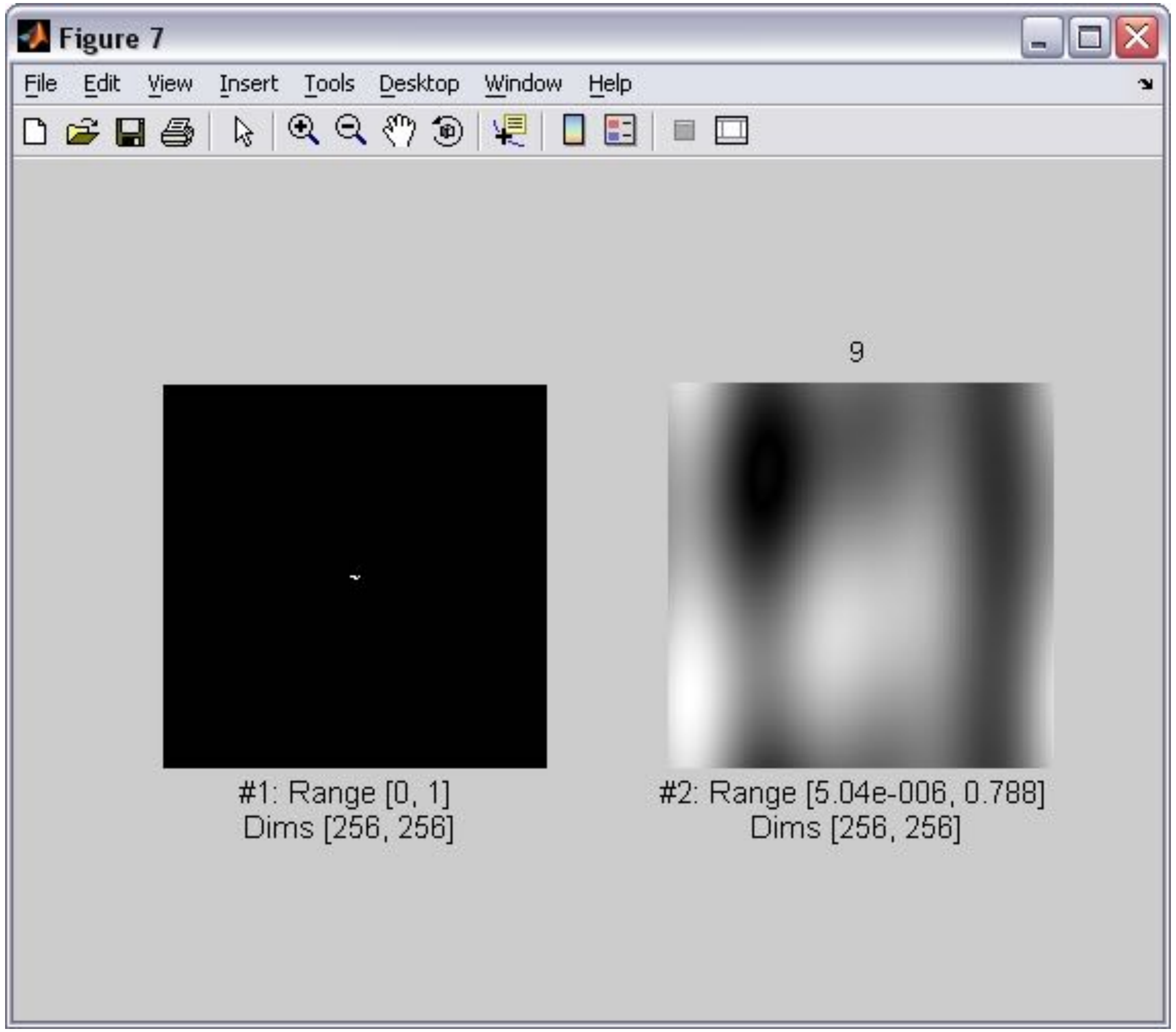


Figure 7
File Edit View Insert Tools Desktop Window Help
9
#1: Range [0, 1]
Dims [256, 256]
#2: Range [5.04e-006, 0.788]
Dims [256, 256]

17

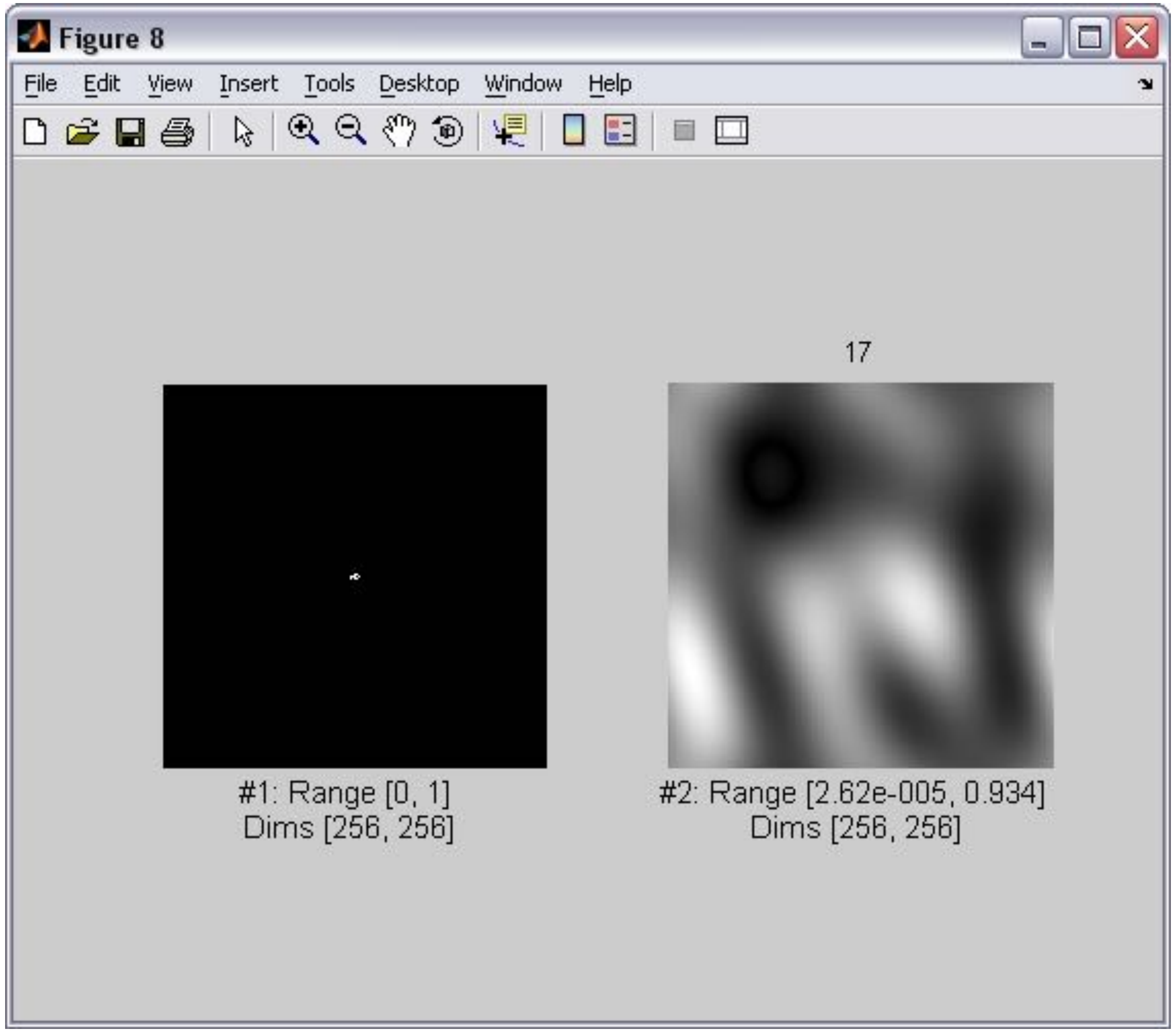


Figure 8
File Edit View Insert Tools Desktop Window Help
17
#1: Range [0, 1]
Dims [256, 256]
#2: Range [2.62e-005, 0.934]
Dims [256, 256]

33

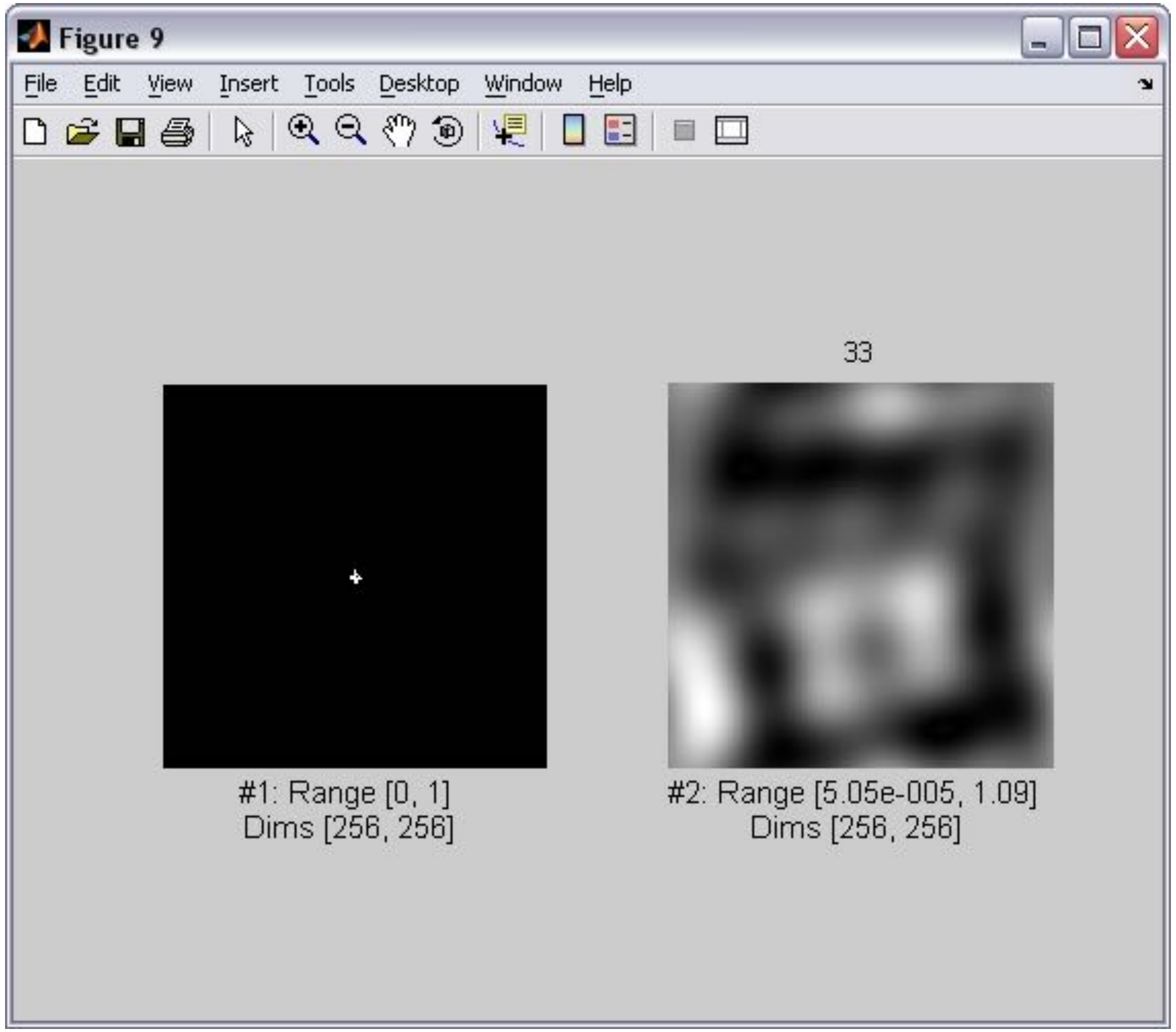


Figure 9
File Edit View Insert Tools Desktop Window Help
33
#1: Range [0, 1]
Dims [256, 256]
#2: Range [5.05e-005, 1.09]
Dims [256, 256]

65

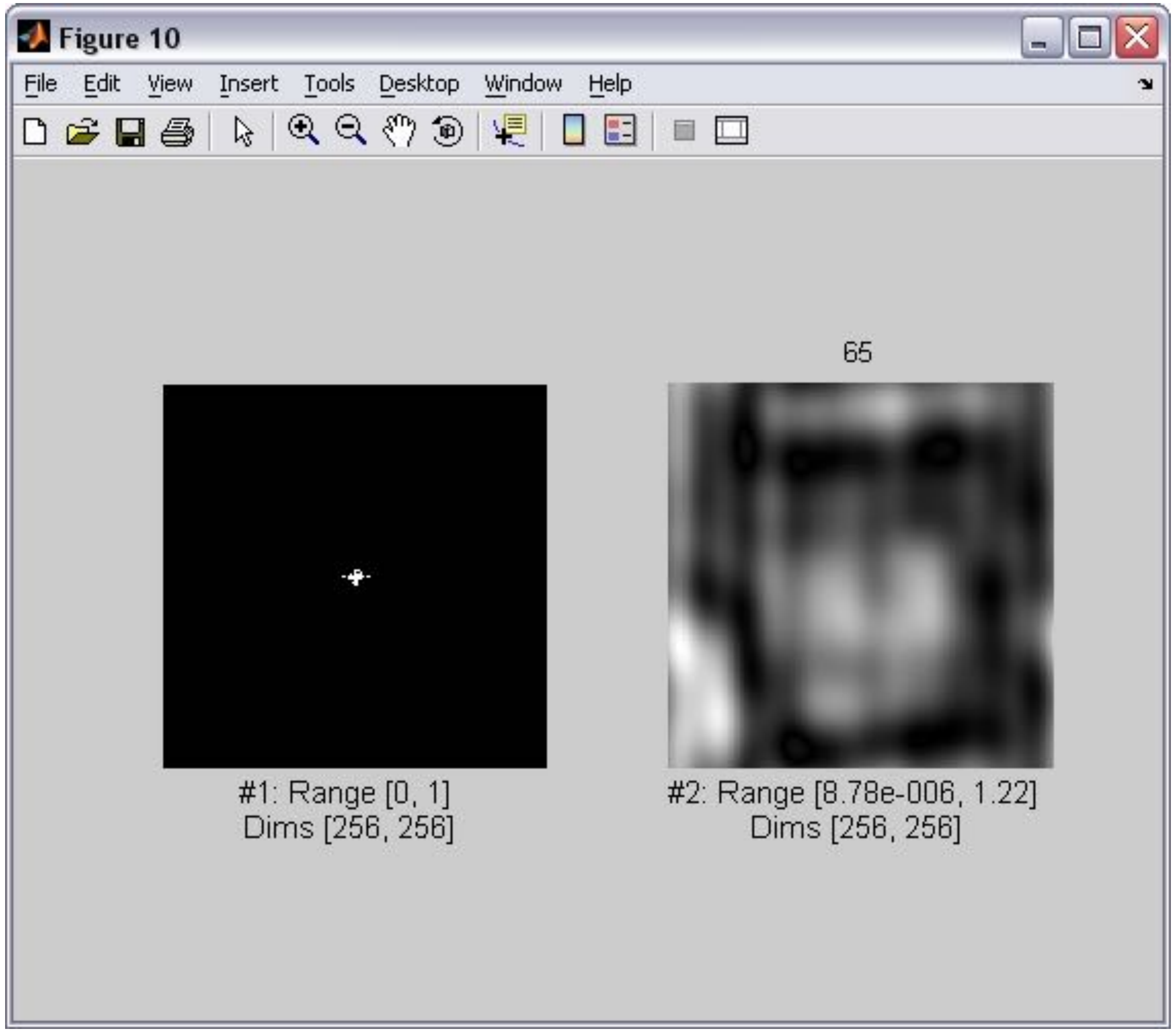


Figure 10
File Edit View Insert Tools Desktop Window Help
65
#1: Range [0, 1]
Dims [256, 256]
#2: Range [8.78e-006, 1.22]
Dims [256, 256]

129

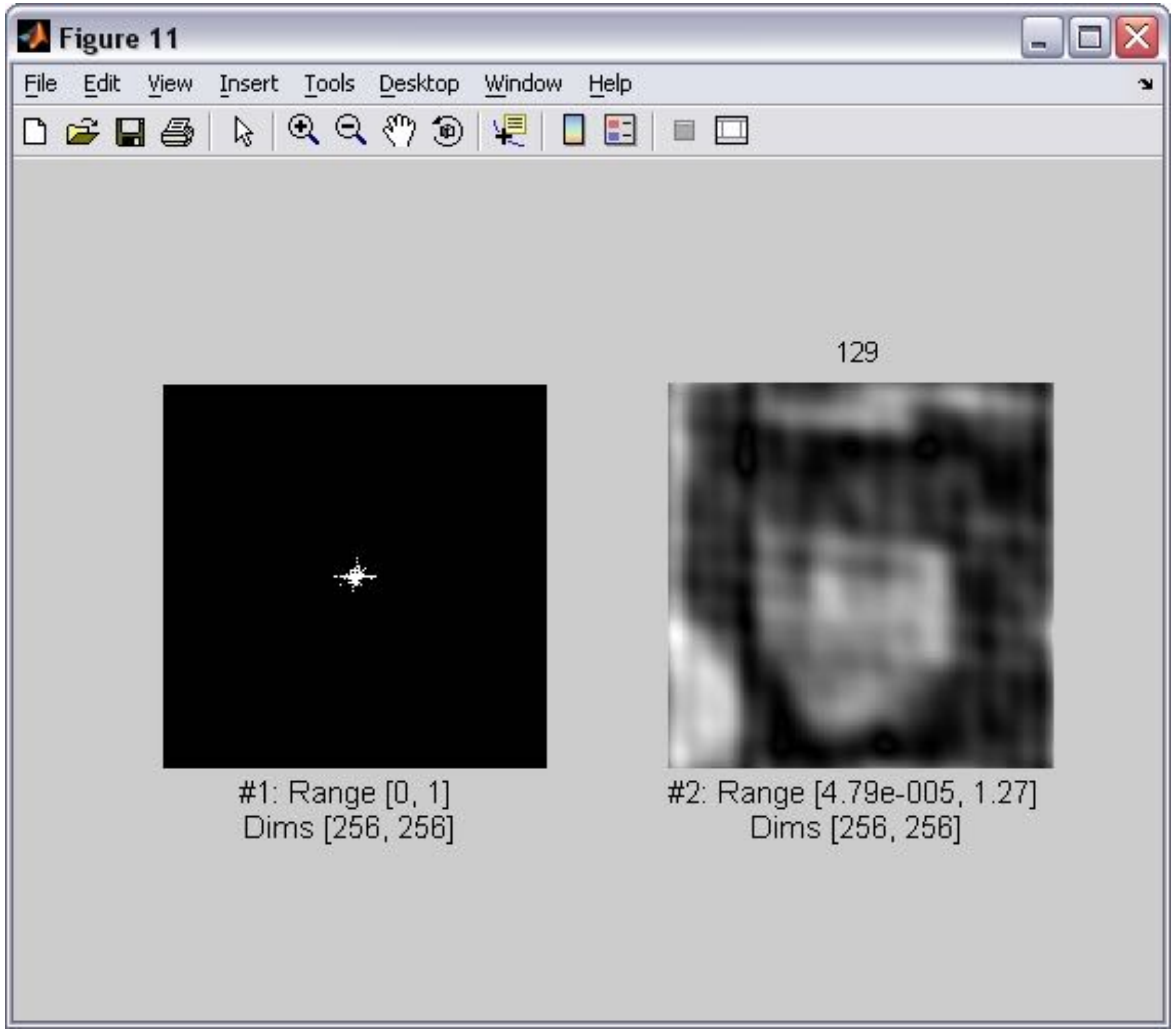


Figure 11
File Edit View Insert Tools Desktop Window Help
129
#1: Range [0, 1]
Dims [256, 256]
#2: Range [4.79e-005, 1.27]
Dims [256, 256]

257

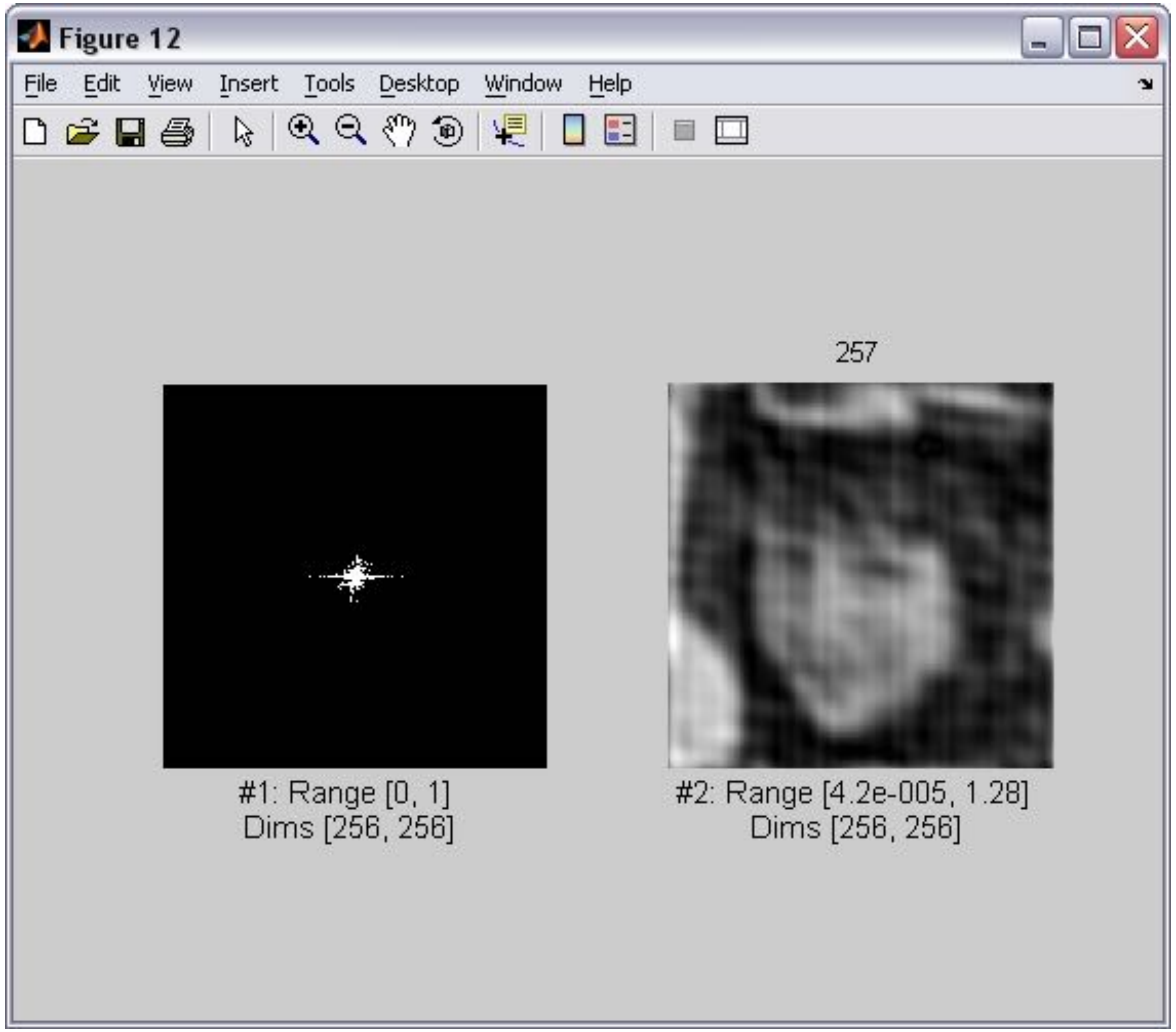


Figure 12
File Edit View Insert Tools Desktop Window Help
257
#1: Range [0, 1]
Dims [256, 256]
#2: Range [4.2e-005, 1.28]
Dims [256, 256]

513

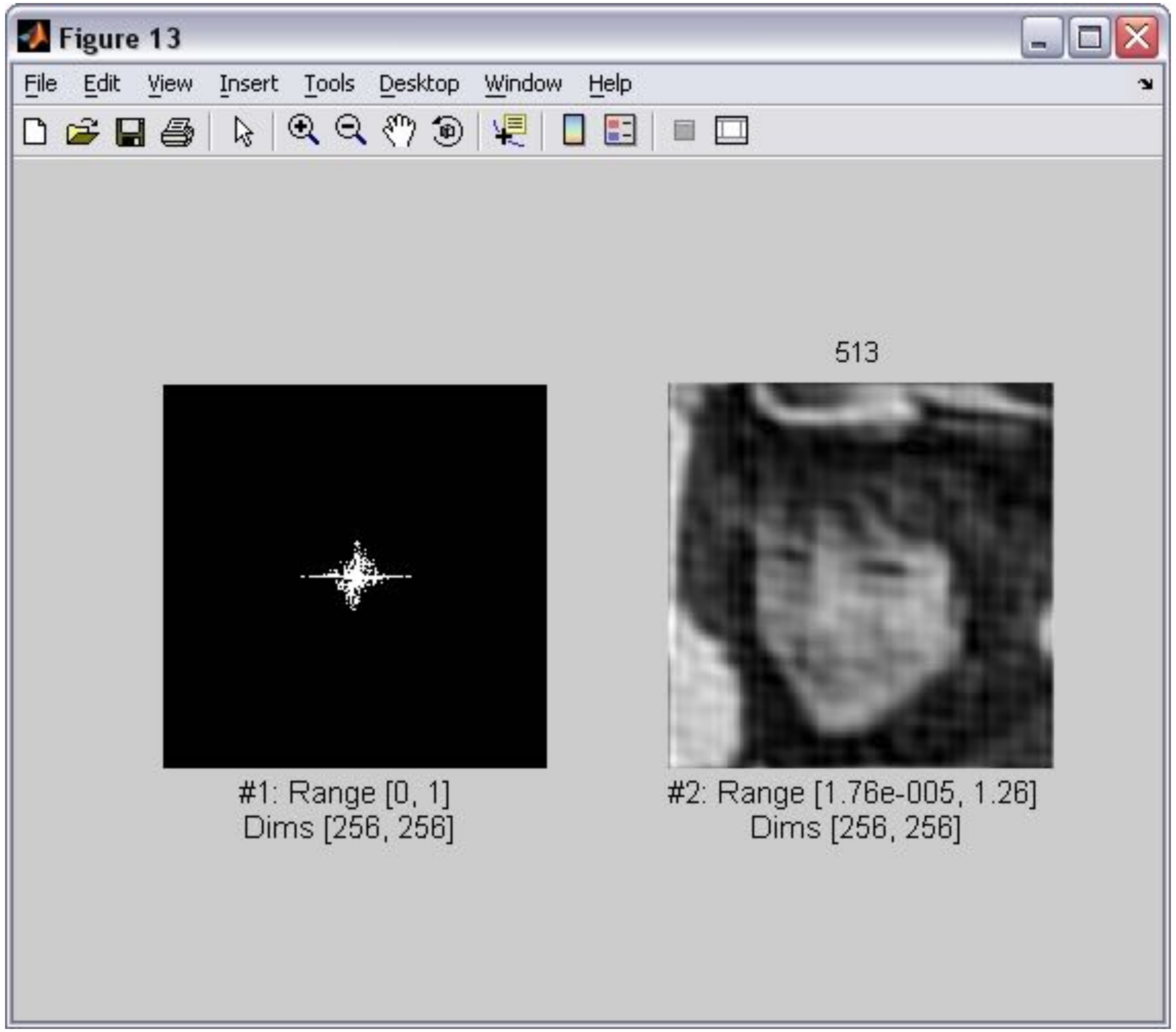


Figure 13
File Edit View Insert Tools Desktop Window Help
513
#1: Range [0, 1]
Dims [256, 256]
#2: Range [1.76e-005, 1.26]
Dims [256, 256]

1025

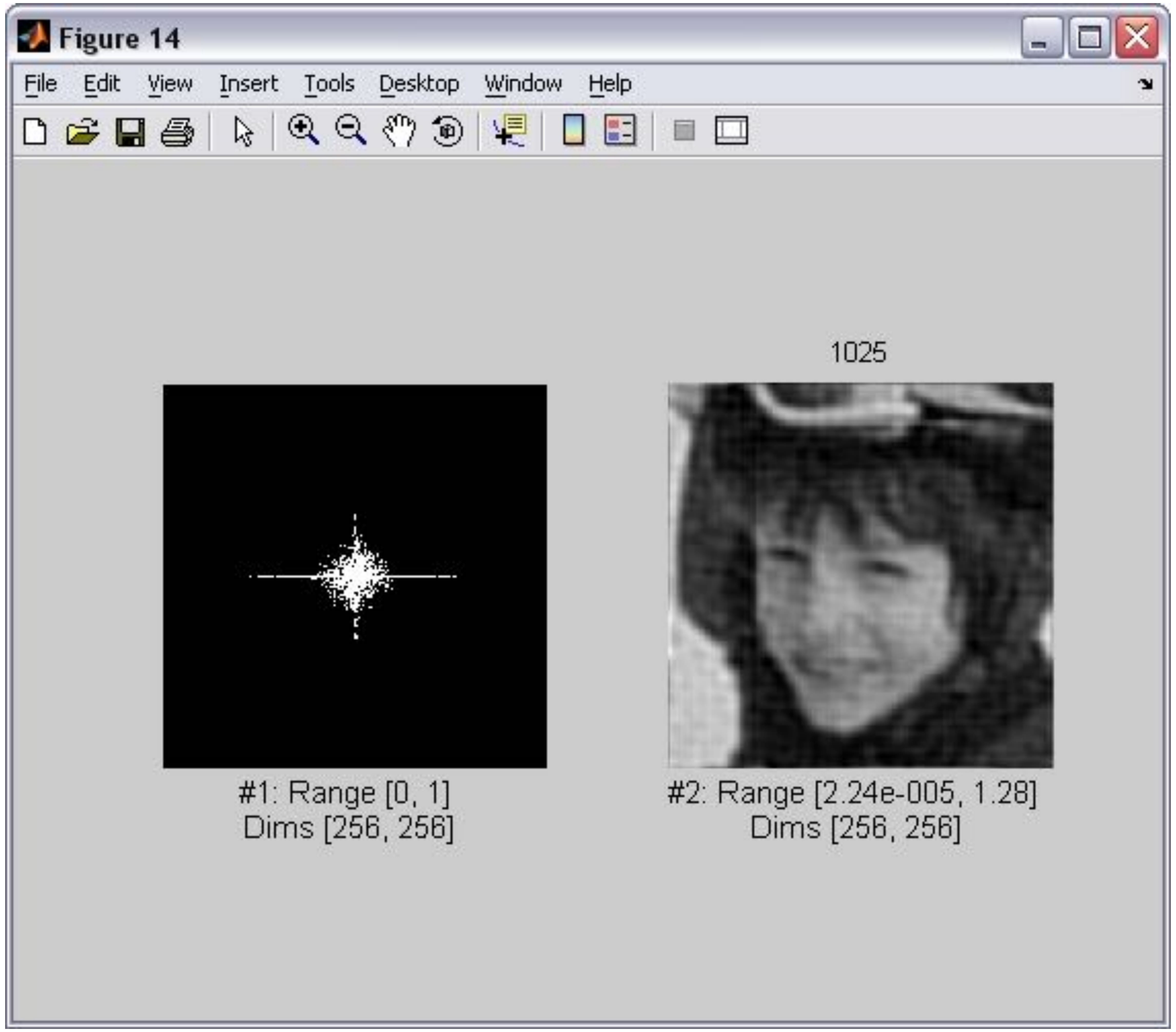


Figure 14
File Edit View Insert Tools Desktop Window Help
1025
#1: Range [0, 1]
Dims [256, 256]
#2: Range [2.24e-005, 1.28]
Dims [256, 256]

2049

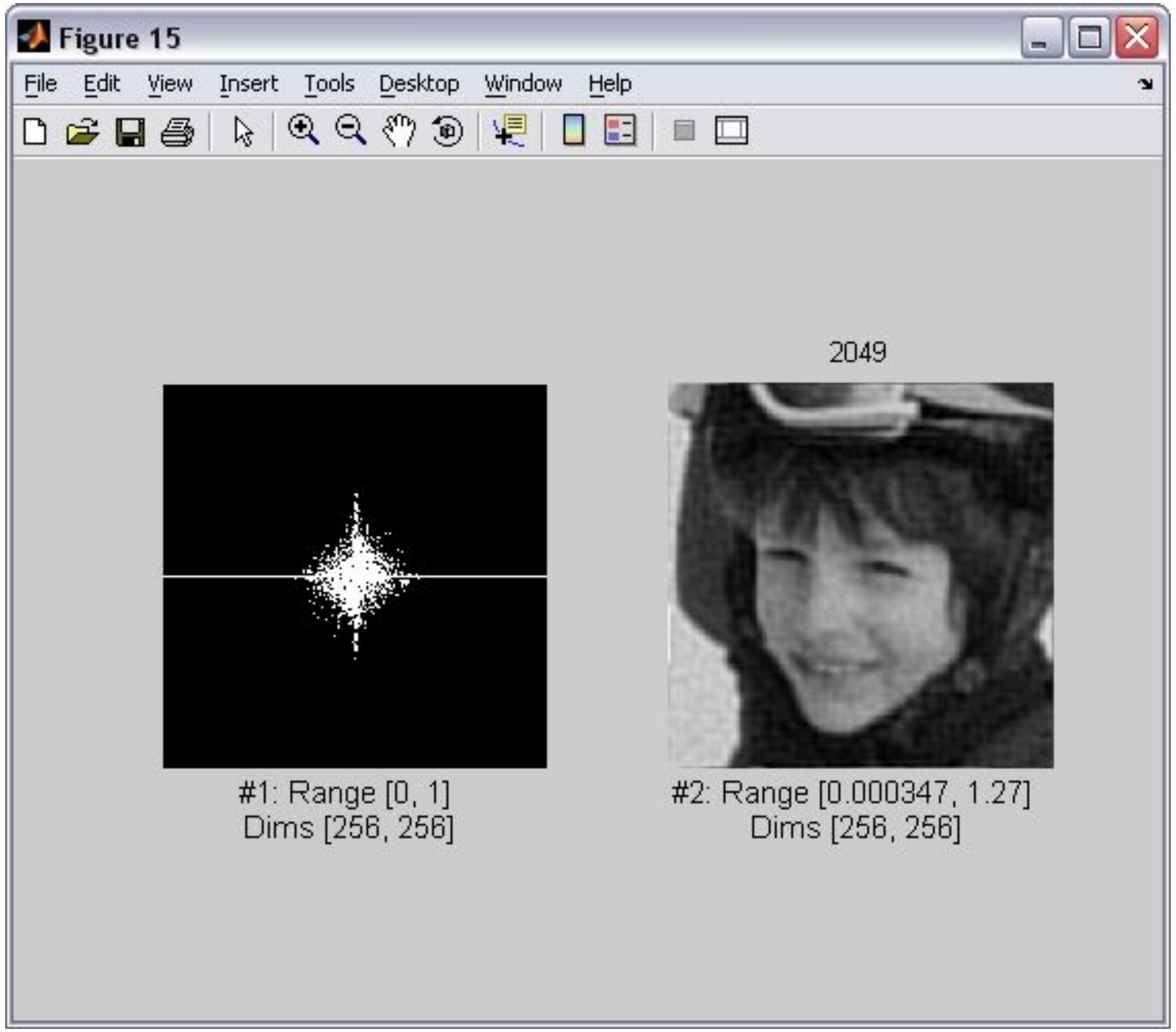


Figure 15
File
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View
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Tools
Desktop
Window
Help
2049
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.000347, 1.27]
Dims [256, 256]

4097

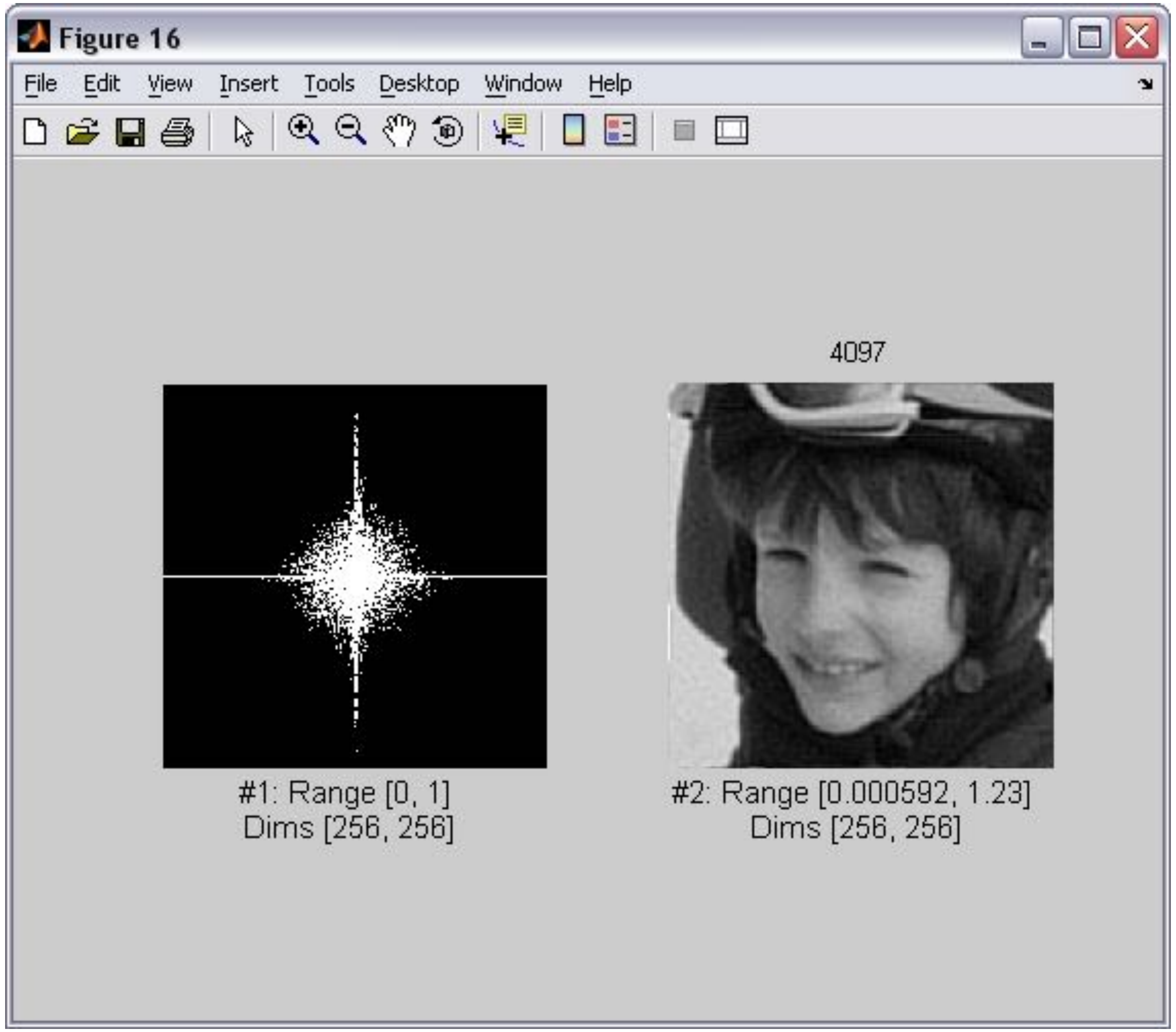


Figure 16
File Edit View Insert Tools Desktop Window Help
4097
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.000592, 1.23]
Dims [256, 256]

8193

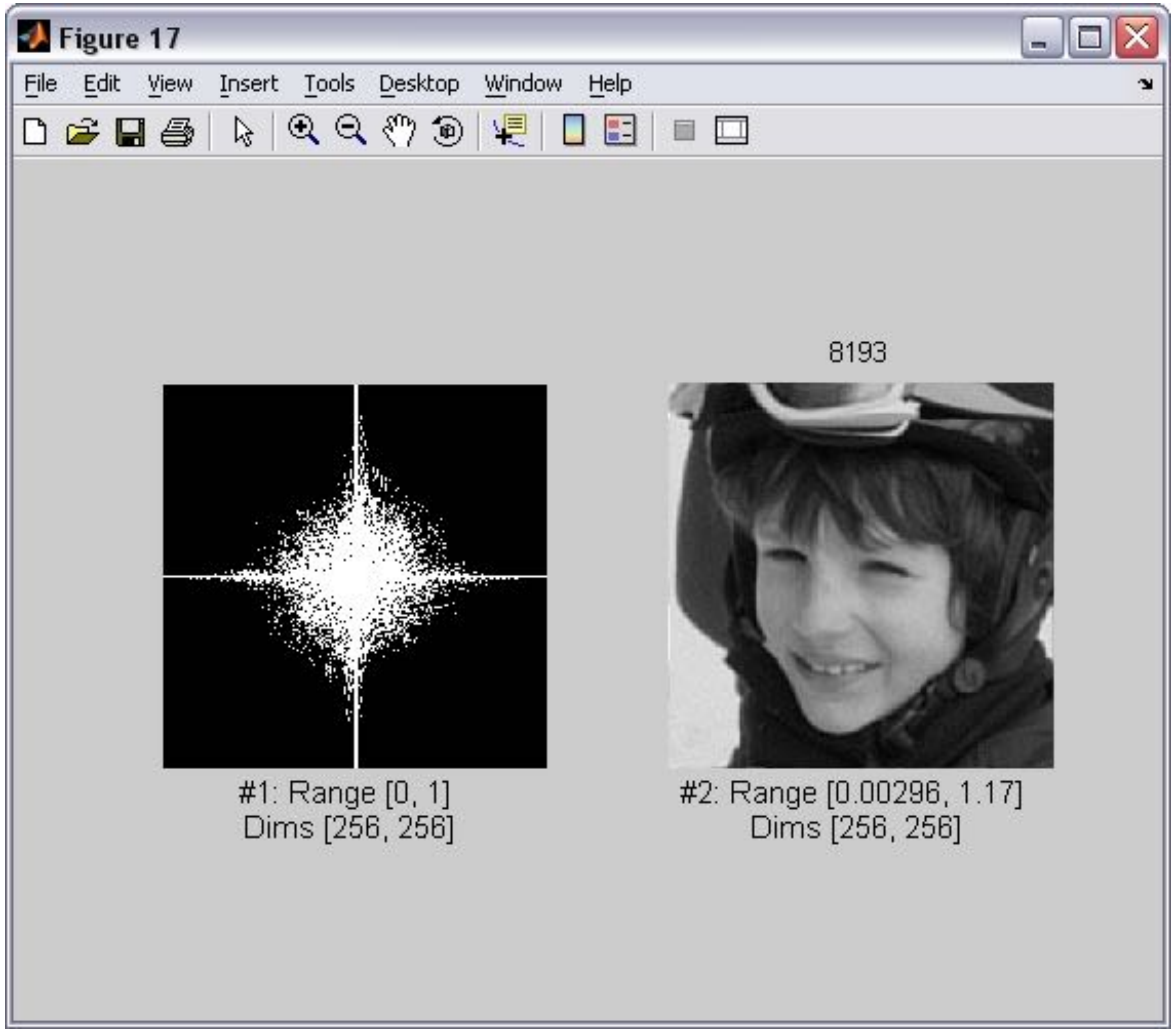


Figure 17
File Edit View Insert Tools Desktop Window Help
8193
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.00296, 1.17]
Dims [256, 256]

16385

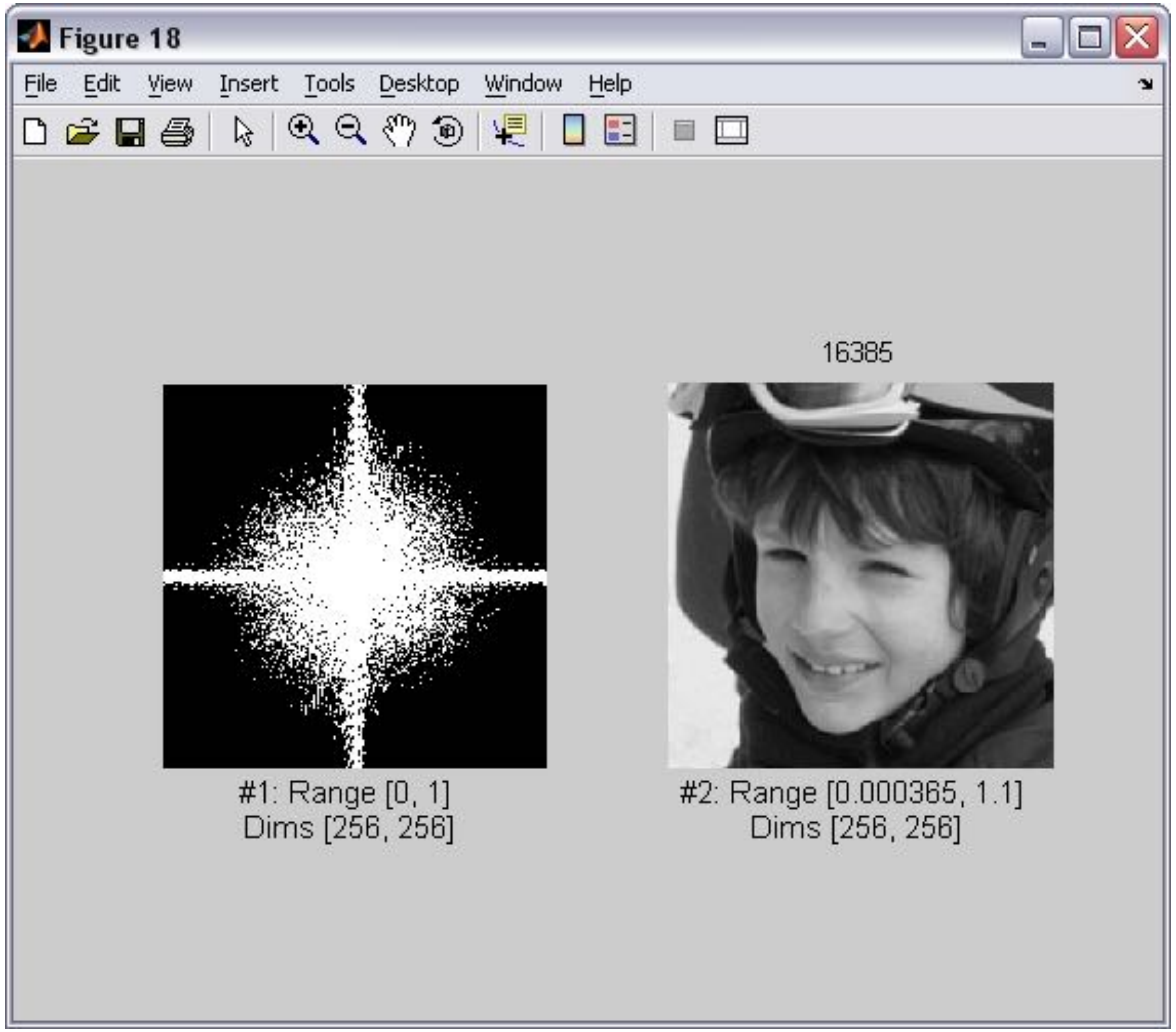


Figure 18
File Edit View Insert Tools Desktop Window Help
16385
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.000365, 1.1]
Dims [256, 256]

32769

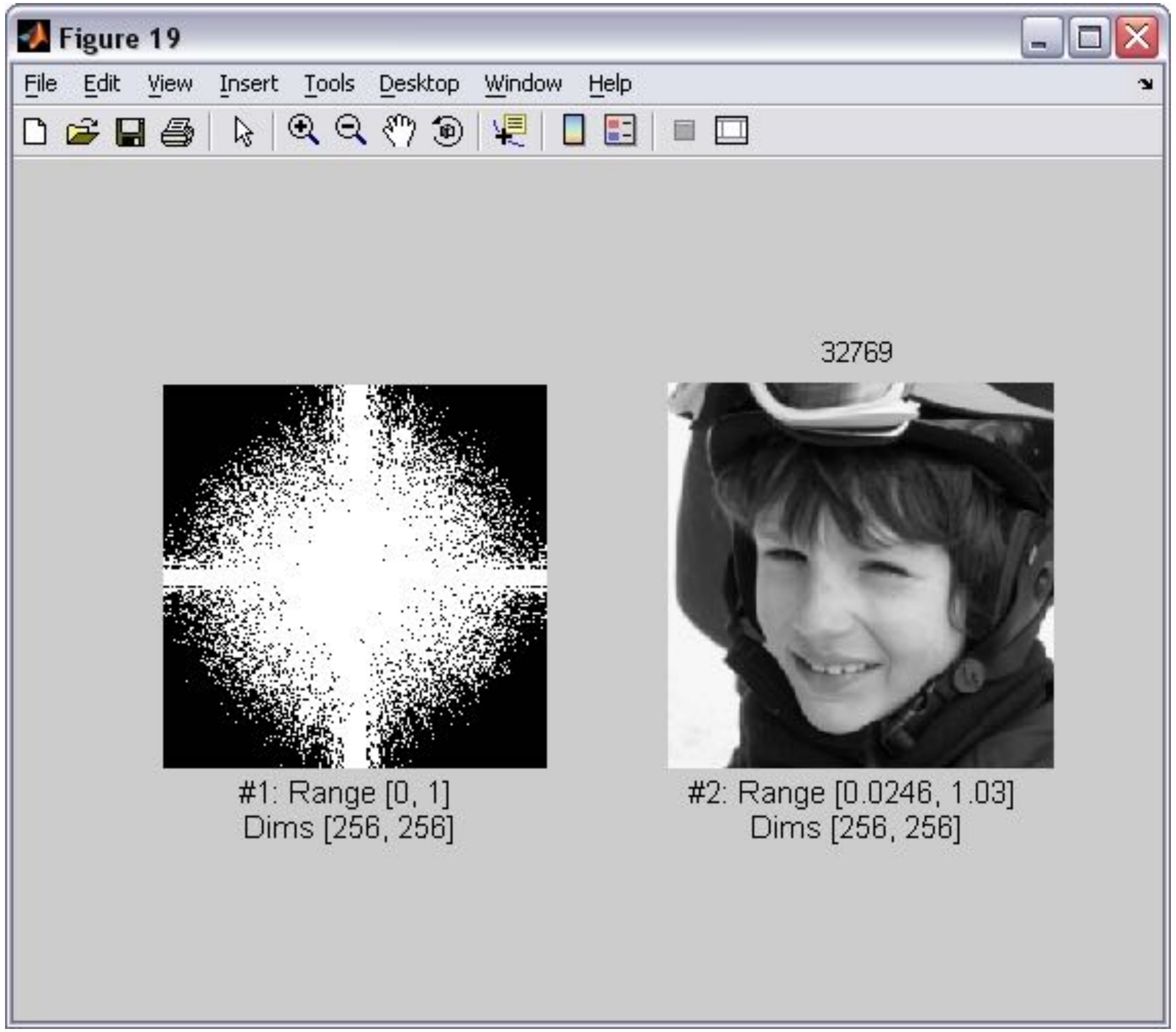


Figure 19
File Edit View Insert Tools Desktop Window Help
32769
#1: Range [0, 1]
Dims [256, 256]
#2: Range [0.0246, 1.03]
Dims [256, 256]

65536

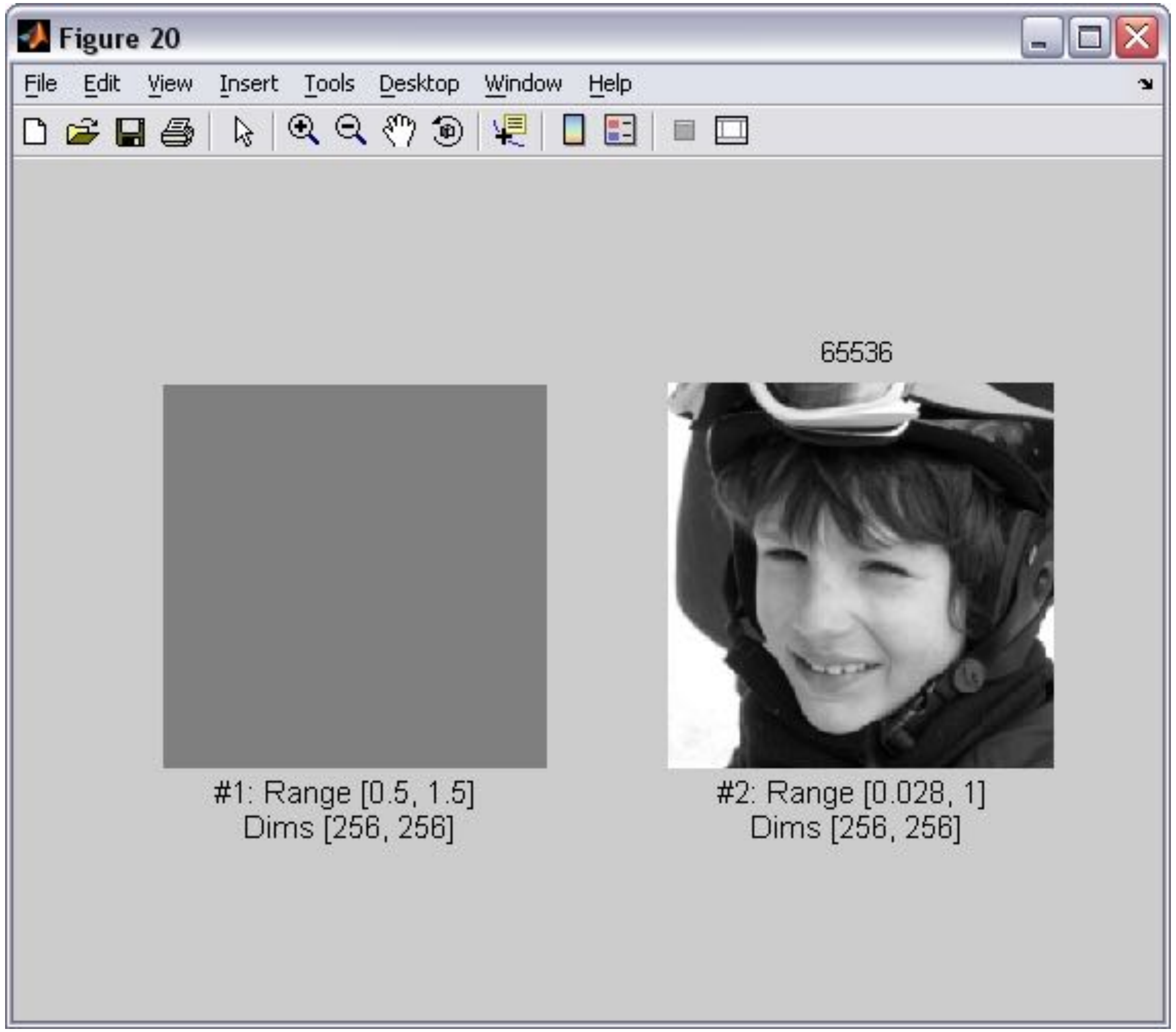
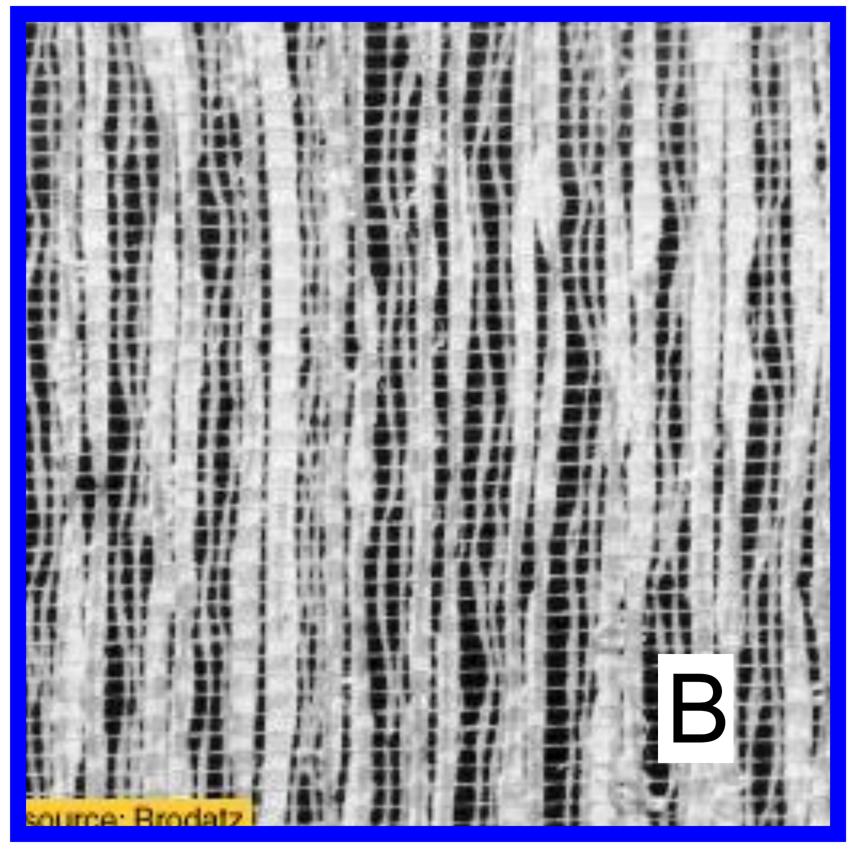
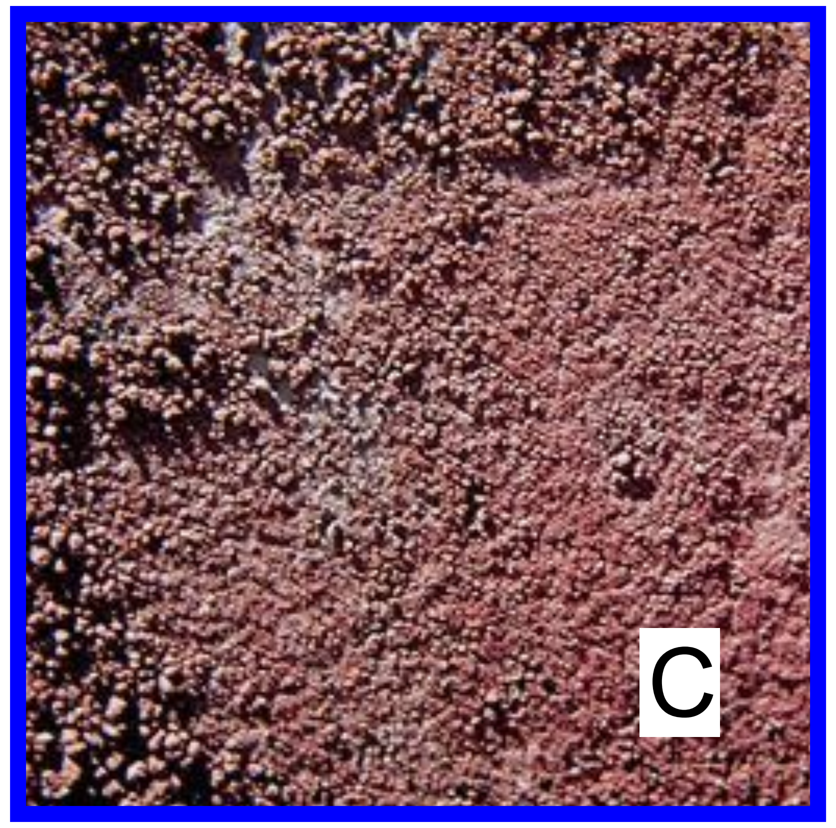


Figure 20
File Edit View Insert Tools Desktop Window Help
65536
#1: Range [0.5, 1.5]
Dims [256, 256]
#2: Range [0.028, 1]
Dims [256, 256]

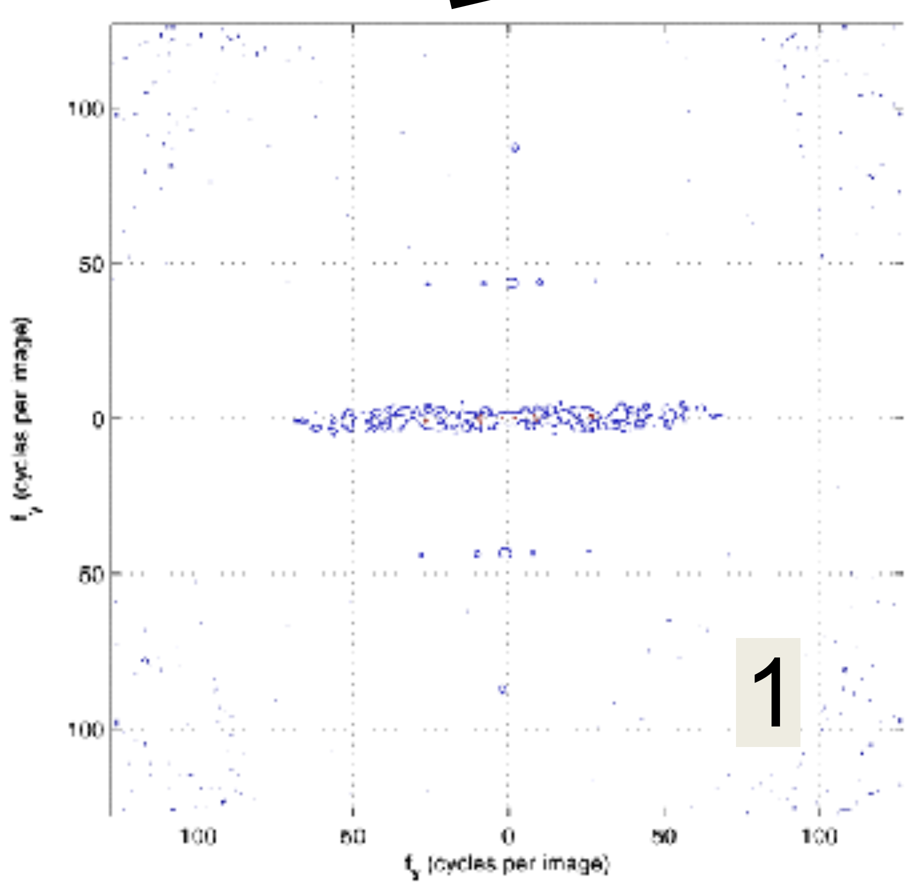
The DFT Game: find the right pairs

Images

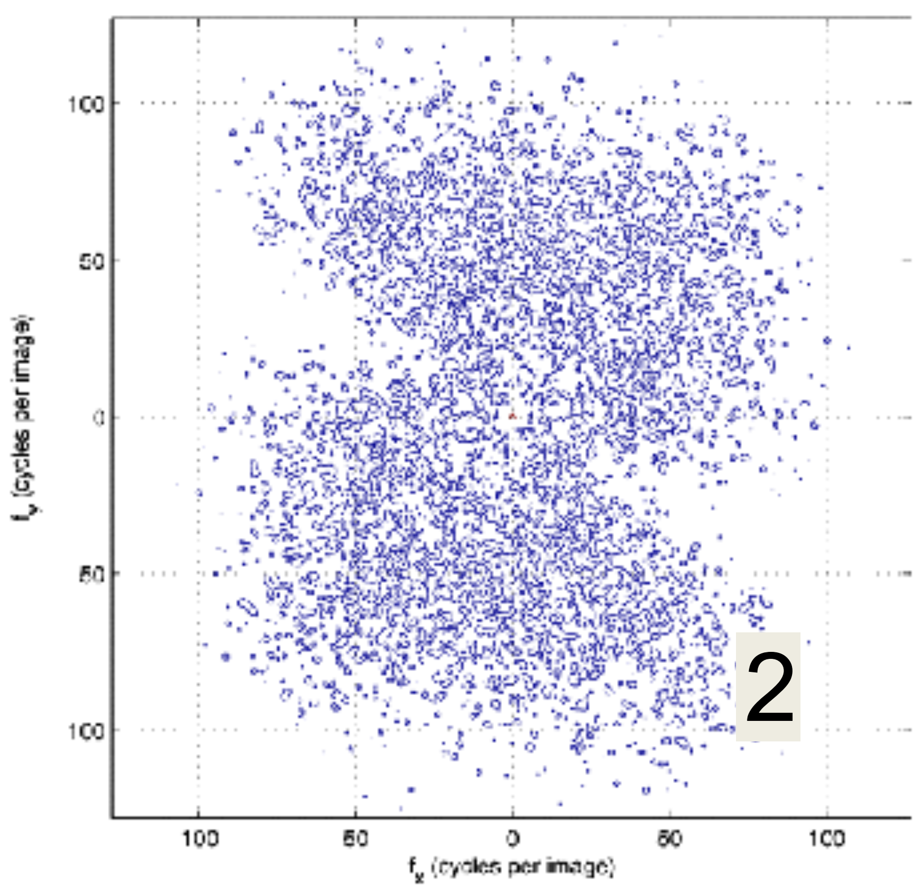




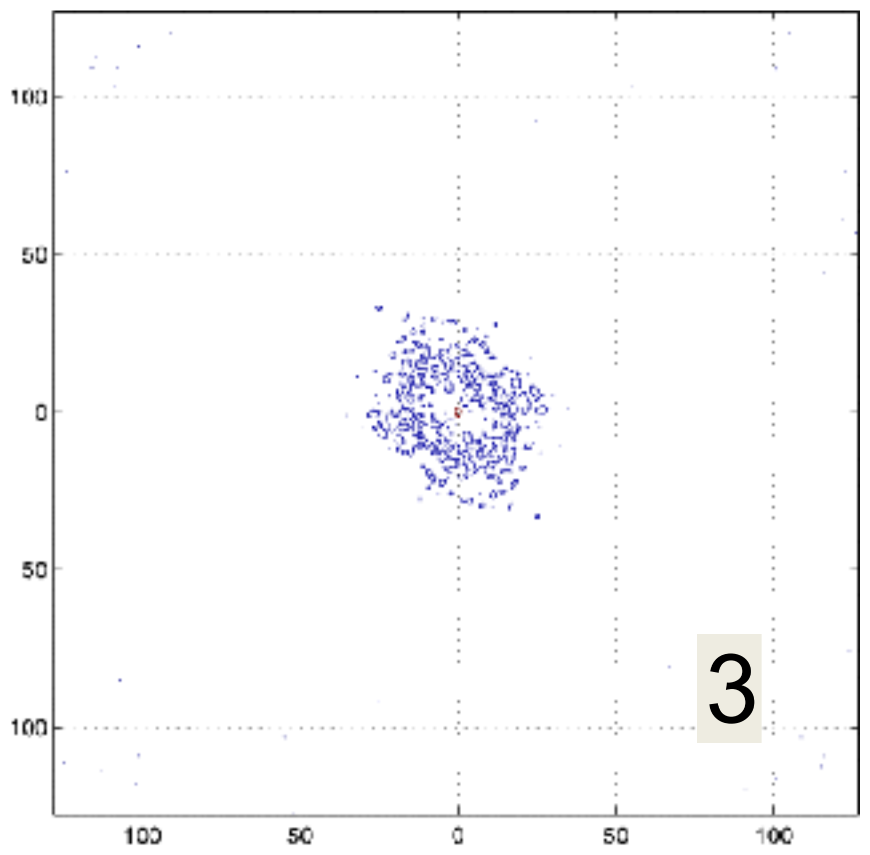
DFT magnitude



fx(cycles/image pixel size)



fx(cycles/image pixel size)

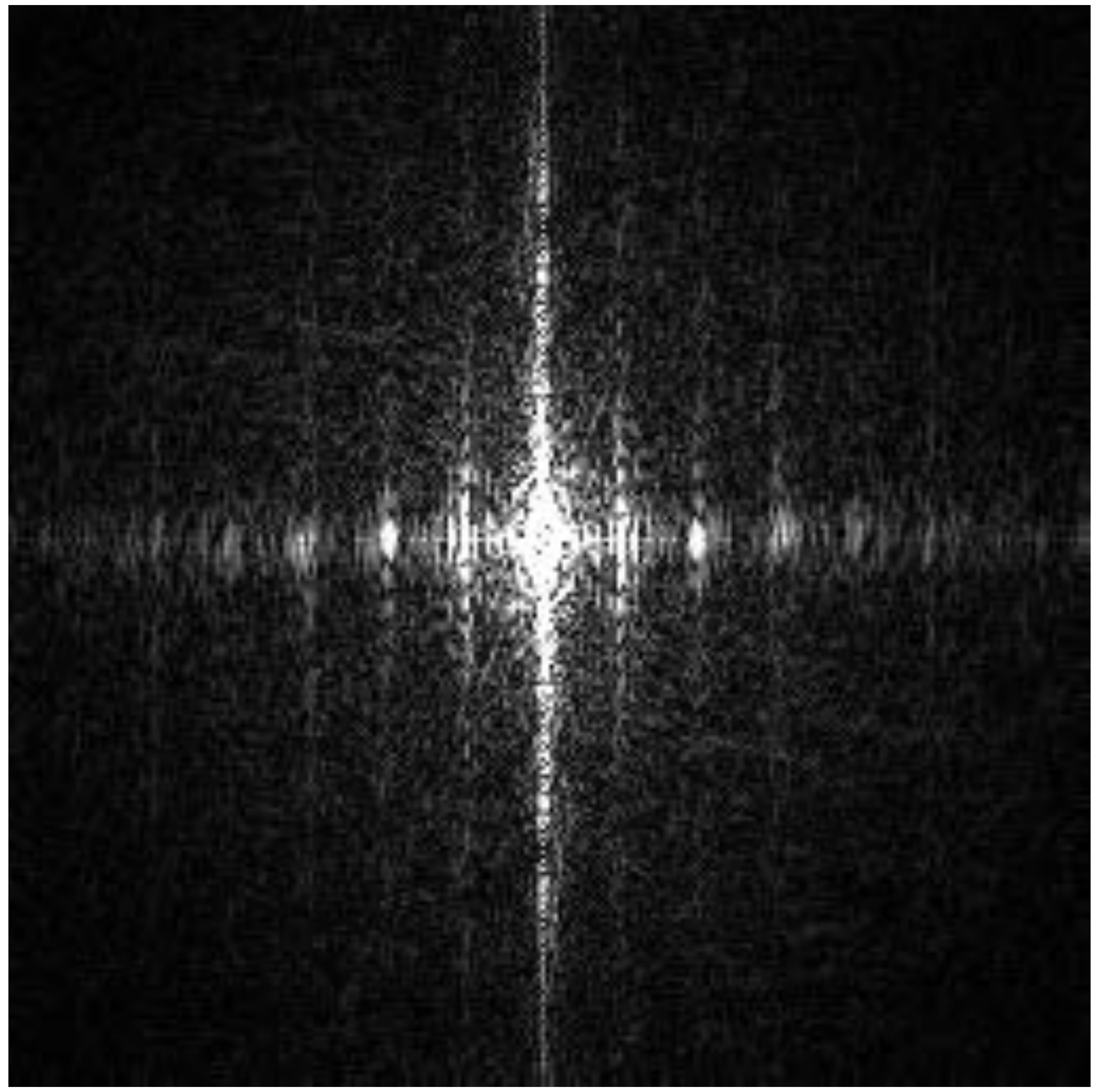


fx(cycles/image pixel size)

The DFT Game: find the right pairs

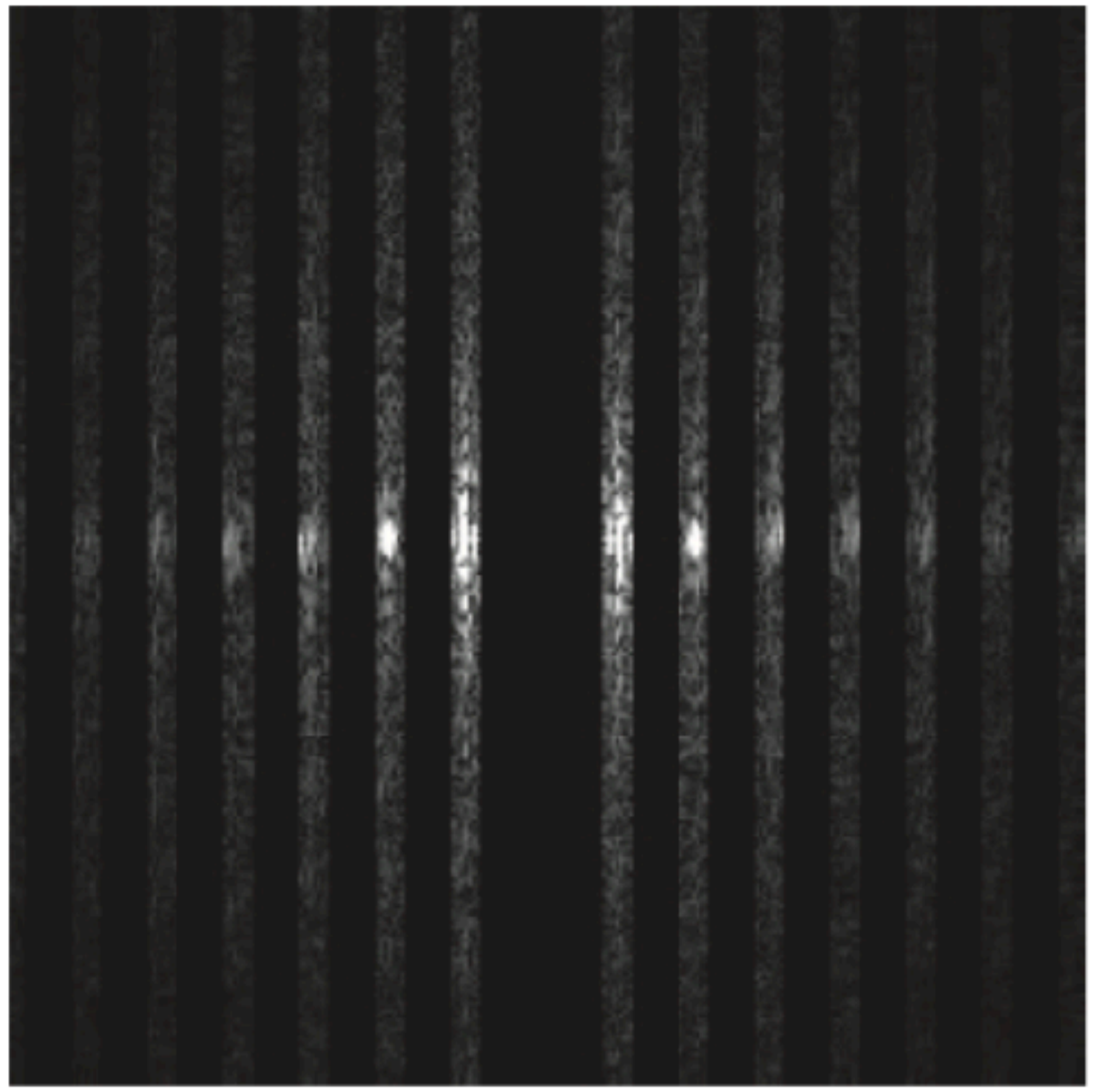
(Solution in the class notes)

DFT $\longrightarrow$

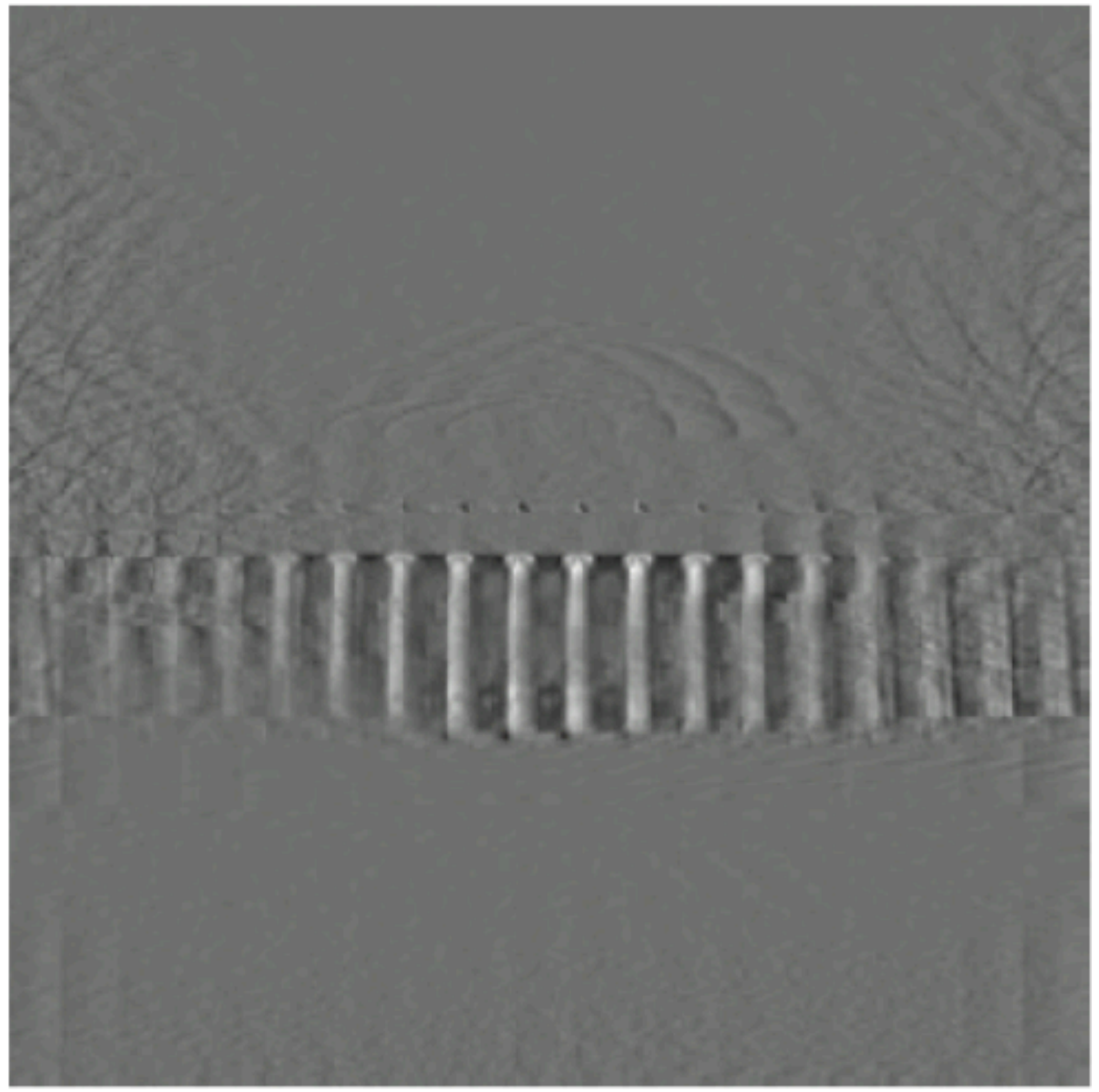


DFT squared magnitude (contrast compressed for visibility)

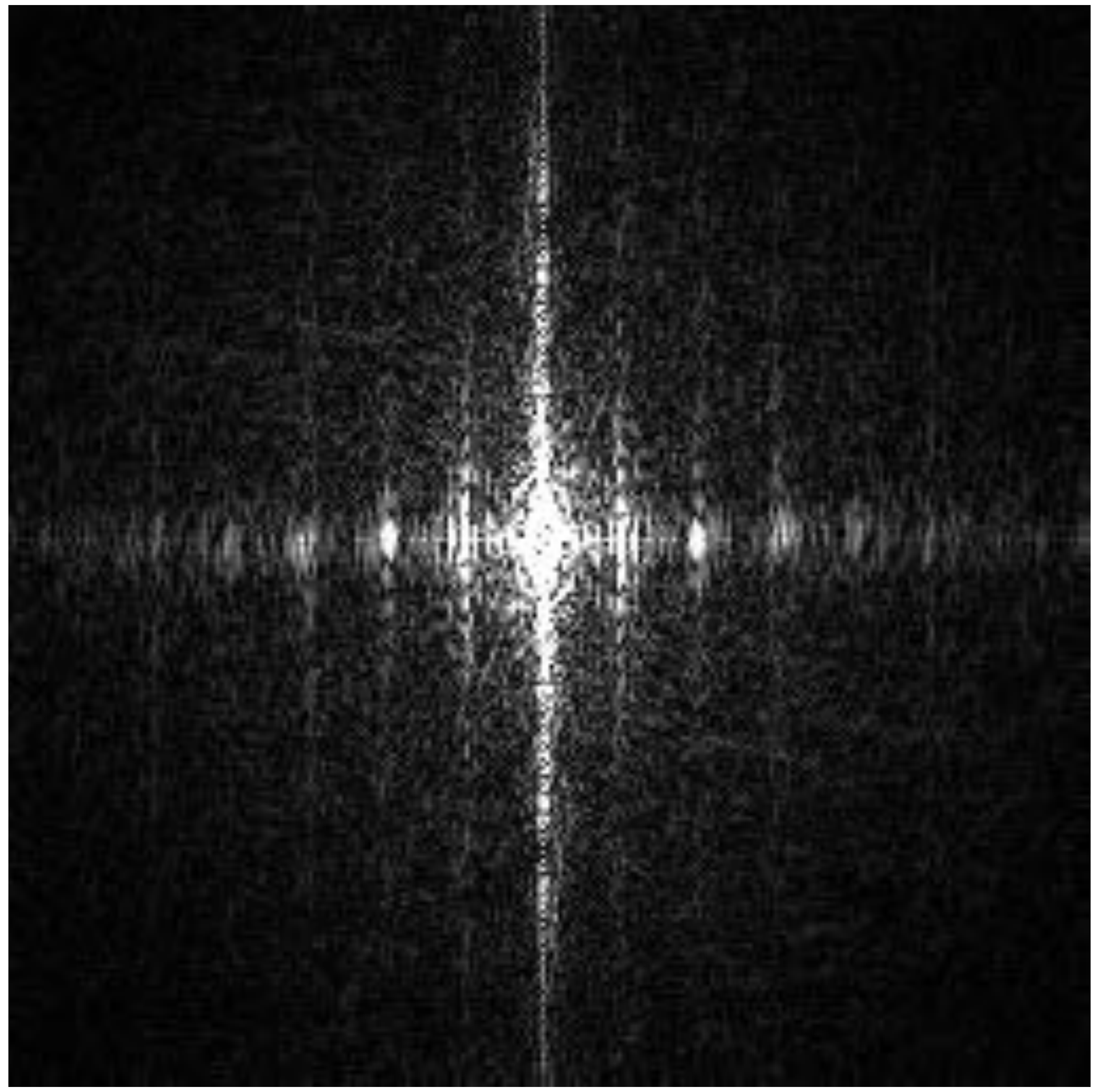
Retain Fourier components corresponding to the repeating columns



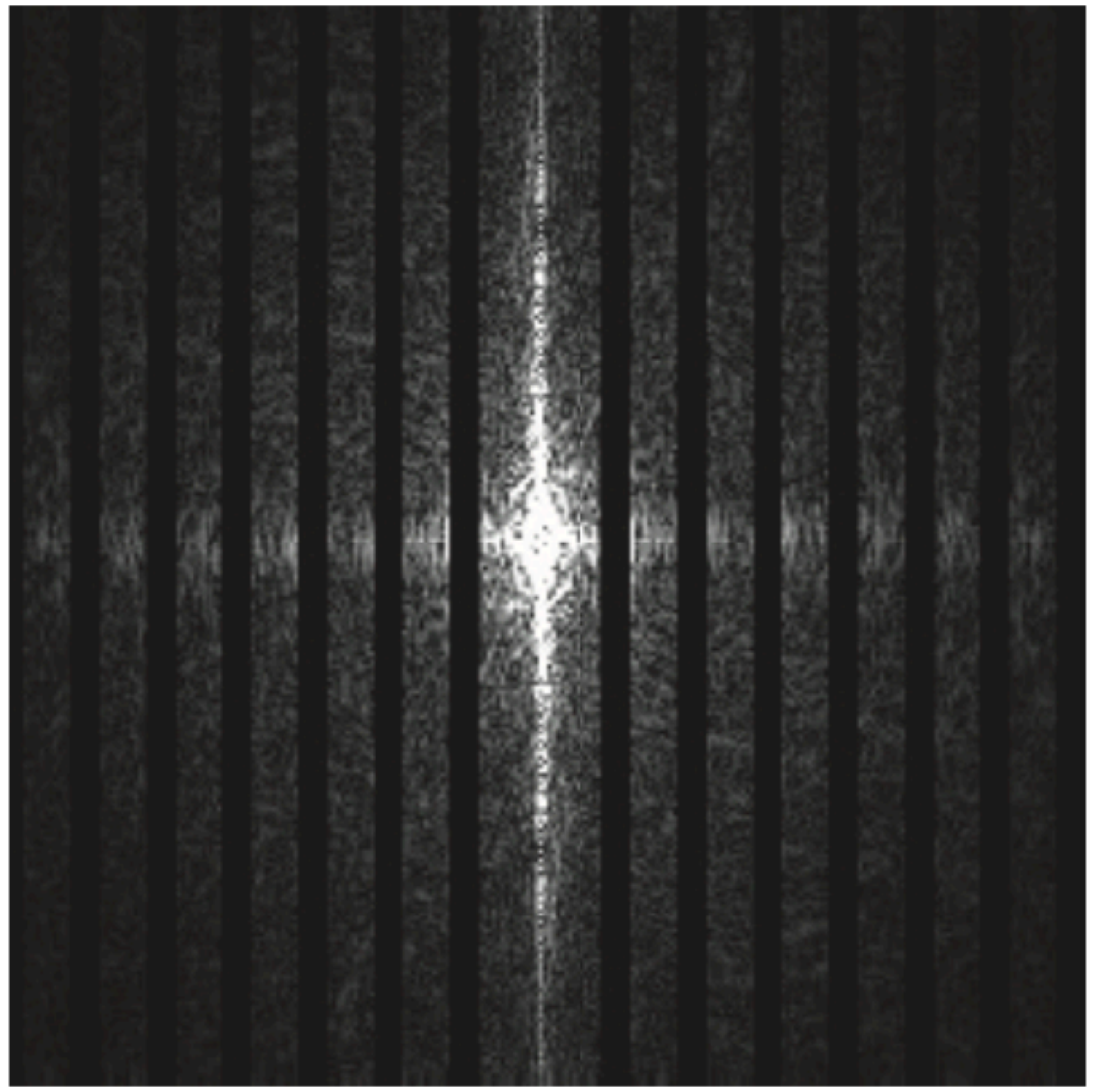
DFT^{-1} $\longrightarrow$



DFT

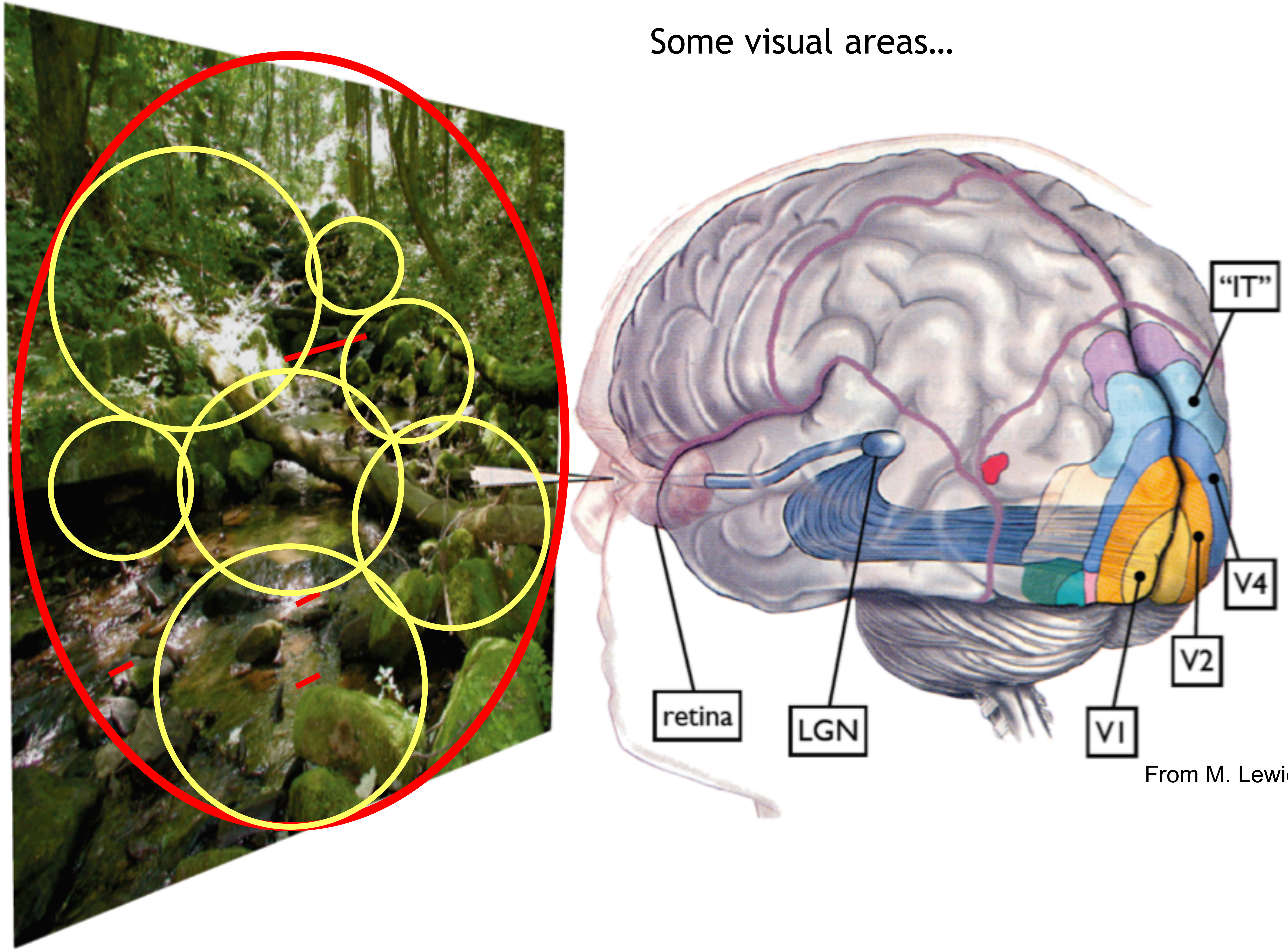


Remove Fourier components corresponding to the repeating columns

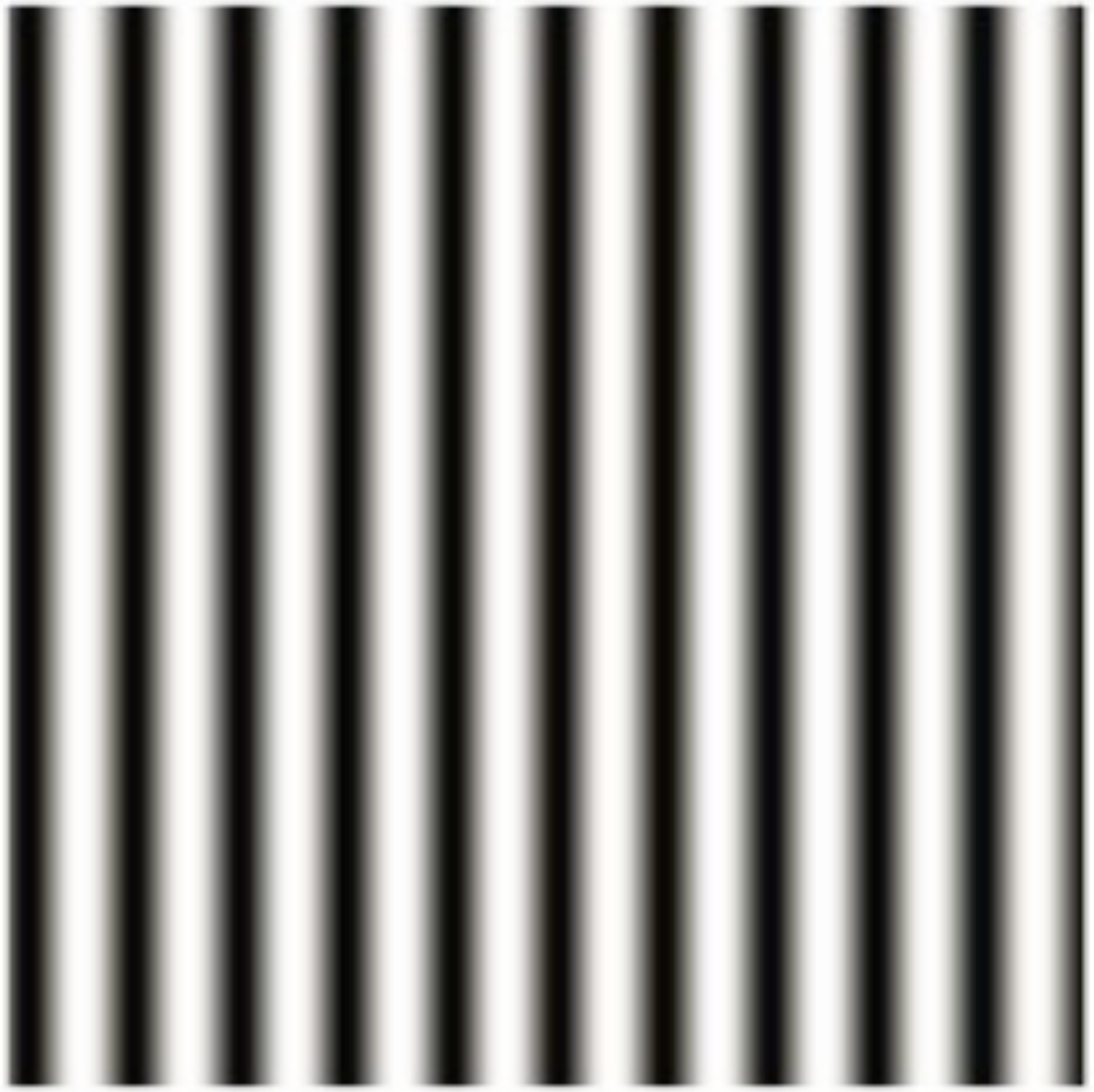


DFT^{-1}

Some visual areas...



From M. Lewicky



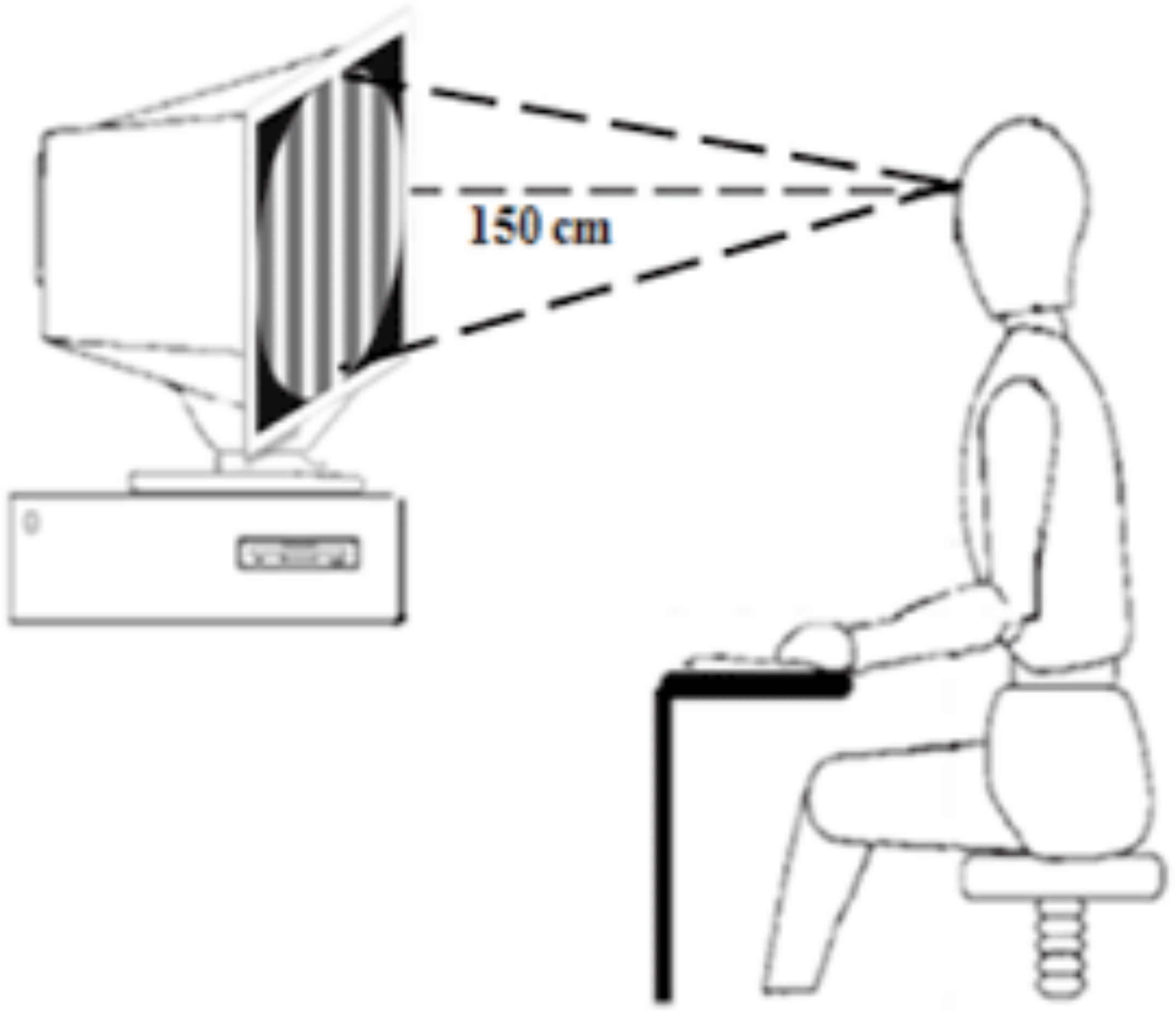
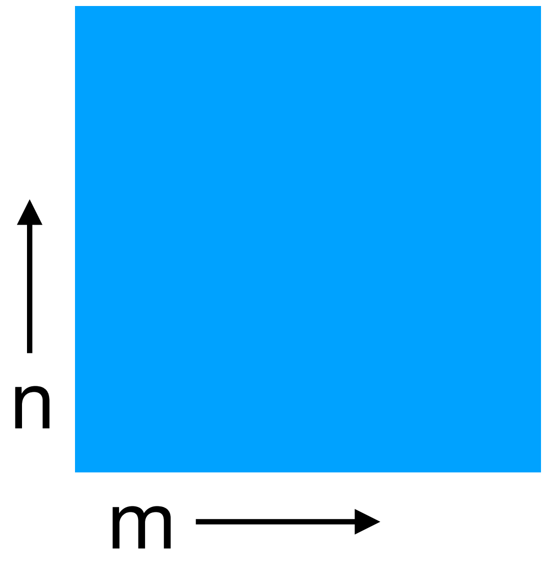


Figure 1. Stimulus presentation scheme. The stimuli were originally calibrated to be seen at a distance of 150 cm in a 19″ display.

Campbell & Robson chart

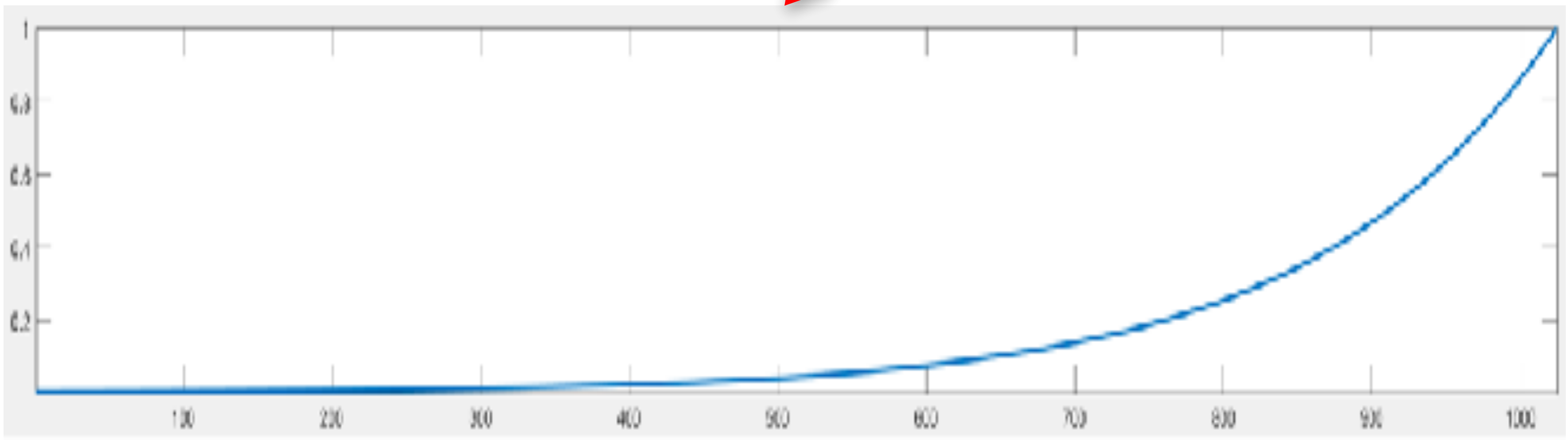


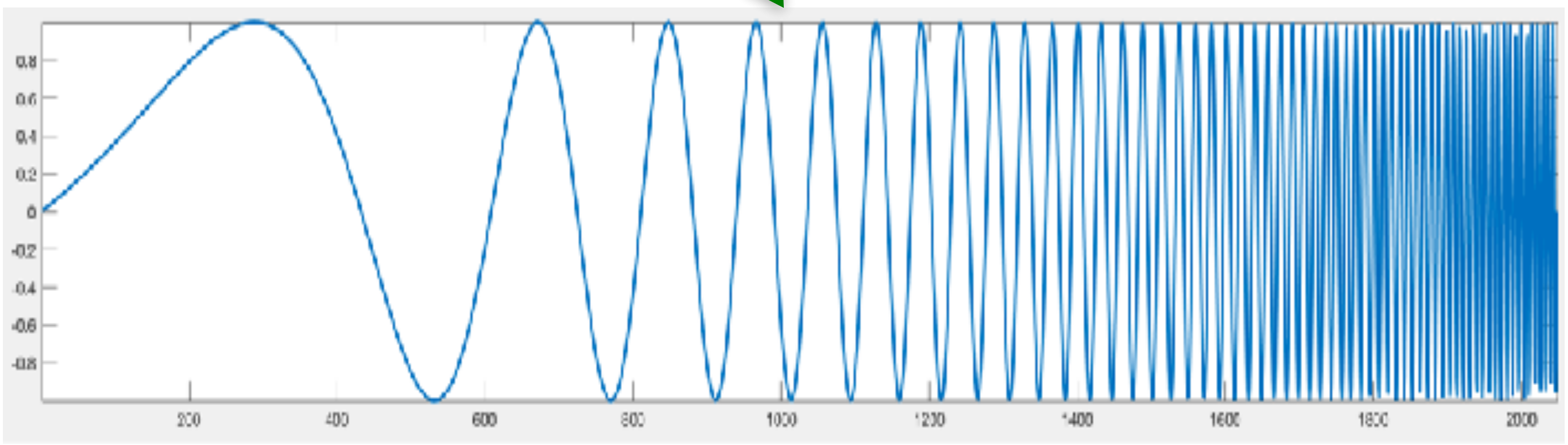
Let's define the following image:

$$\mathbf{I}[n,m] = \boxed{A[n]}\,\boxed{\sin(2\pi f[m]\, m/M)}$$

With:

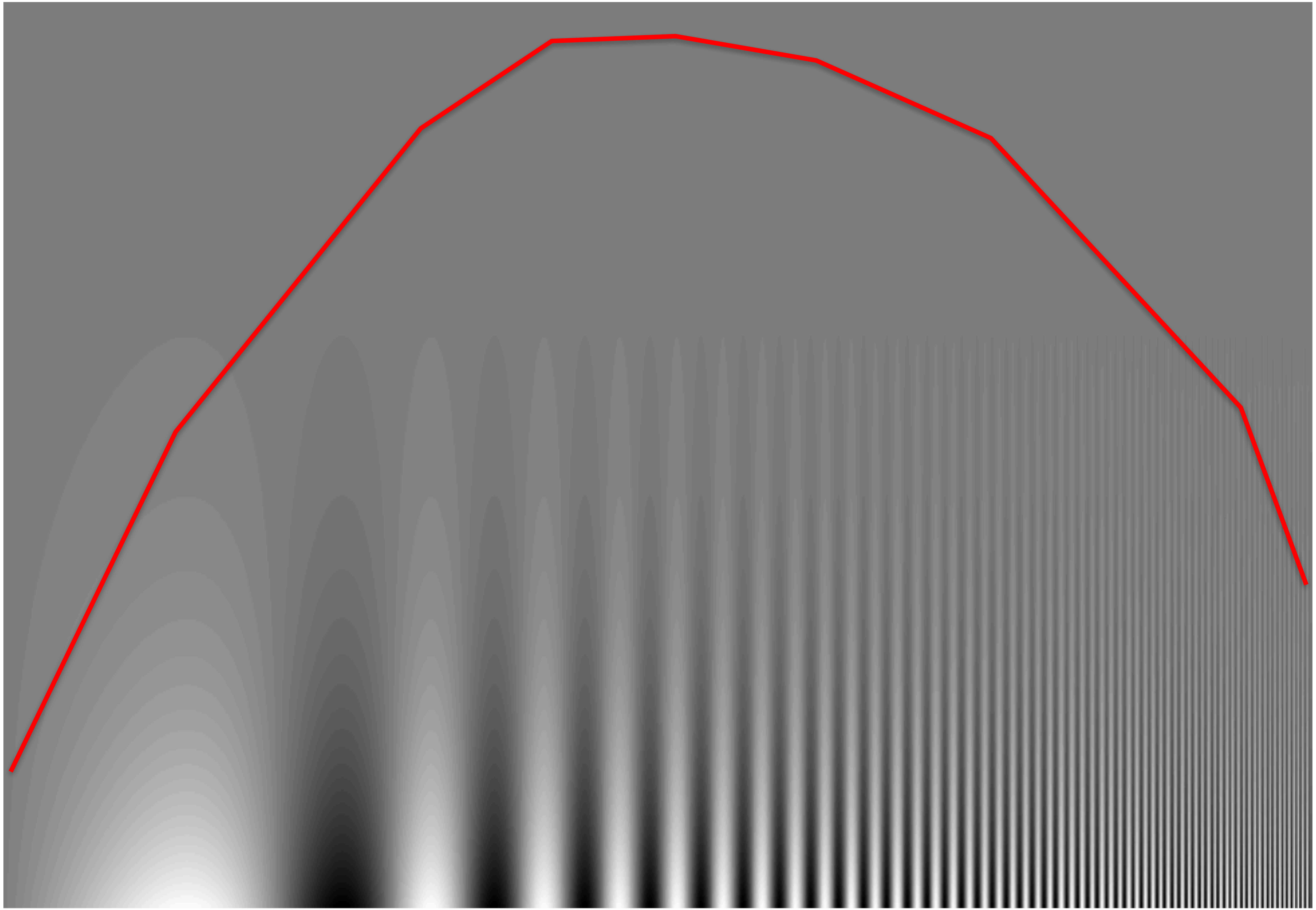
$$A[n] = A_{min}\left(\frac{A_{max}}{A_{min}}\right)^{n/N} \qquad f[m] = f_{min}\left(\frac{f_{max}}{f_{min}}\right)^{m/M}$$





What do you think you should see when looking at this image?

$$\mathbf{I}[n,m] = A[n] \sin(2\pi f[m]\, m/M)$$

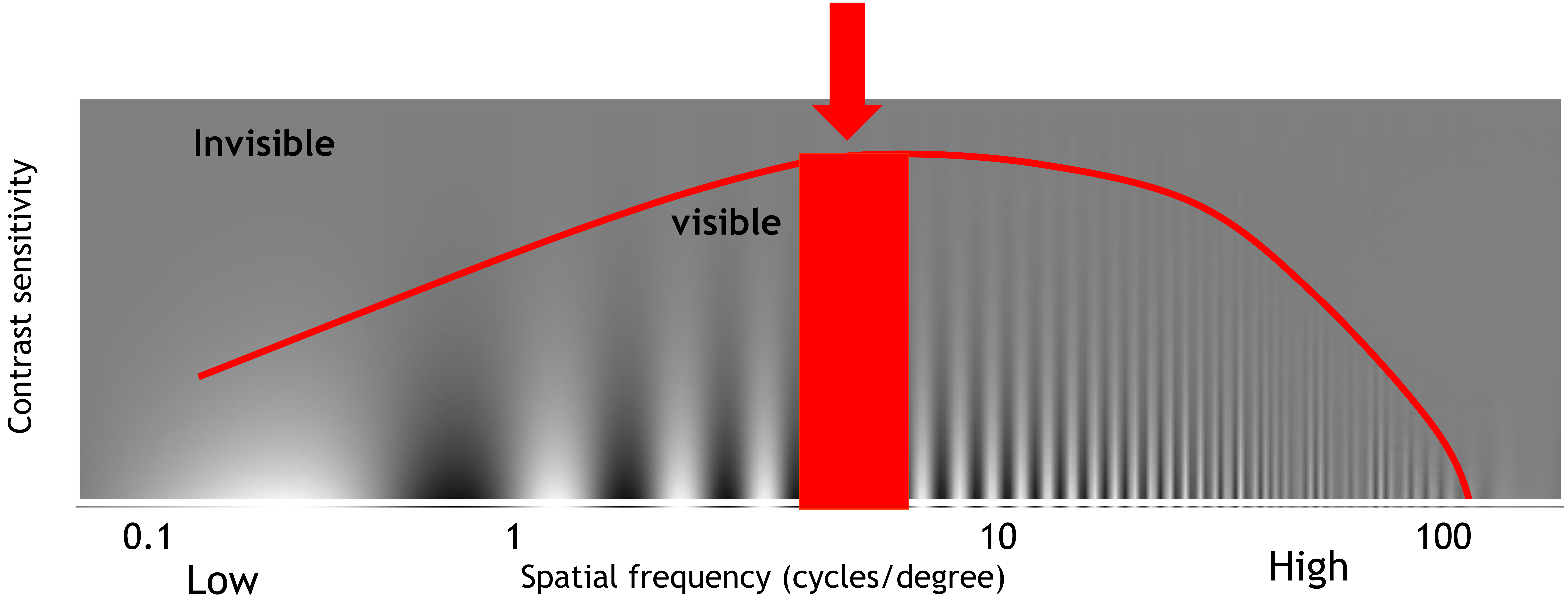


Contrast Sensitivity Function

Blackmore & Campbell (1969)

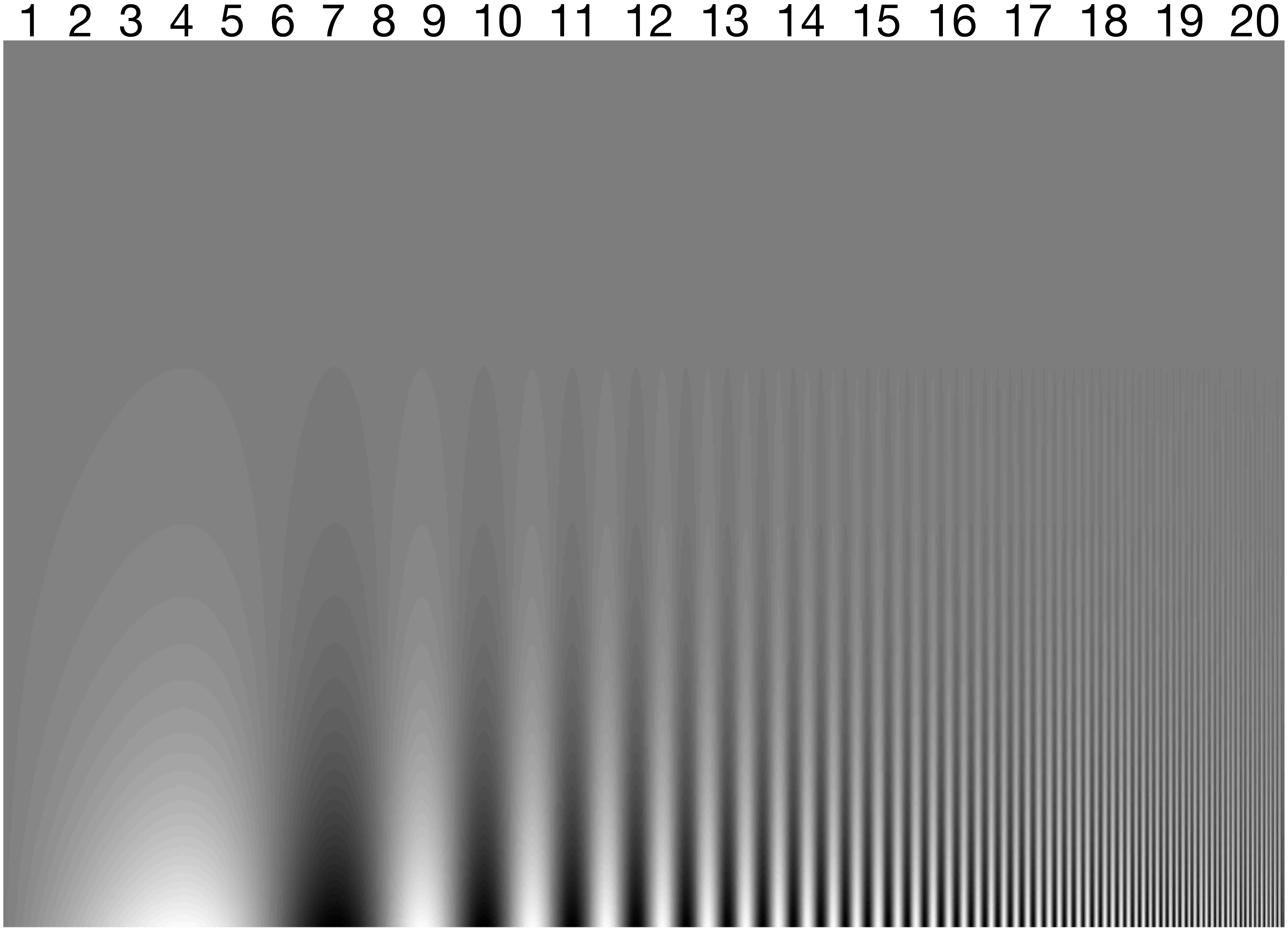
Maximum sensitivity

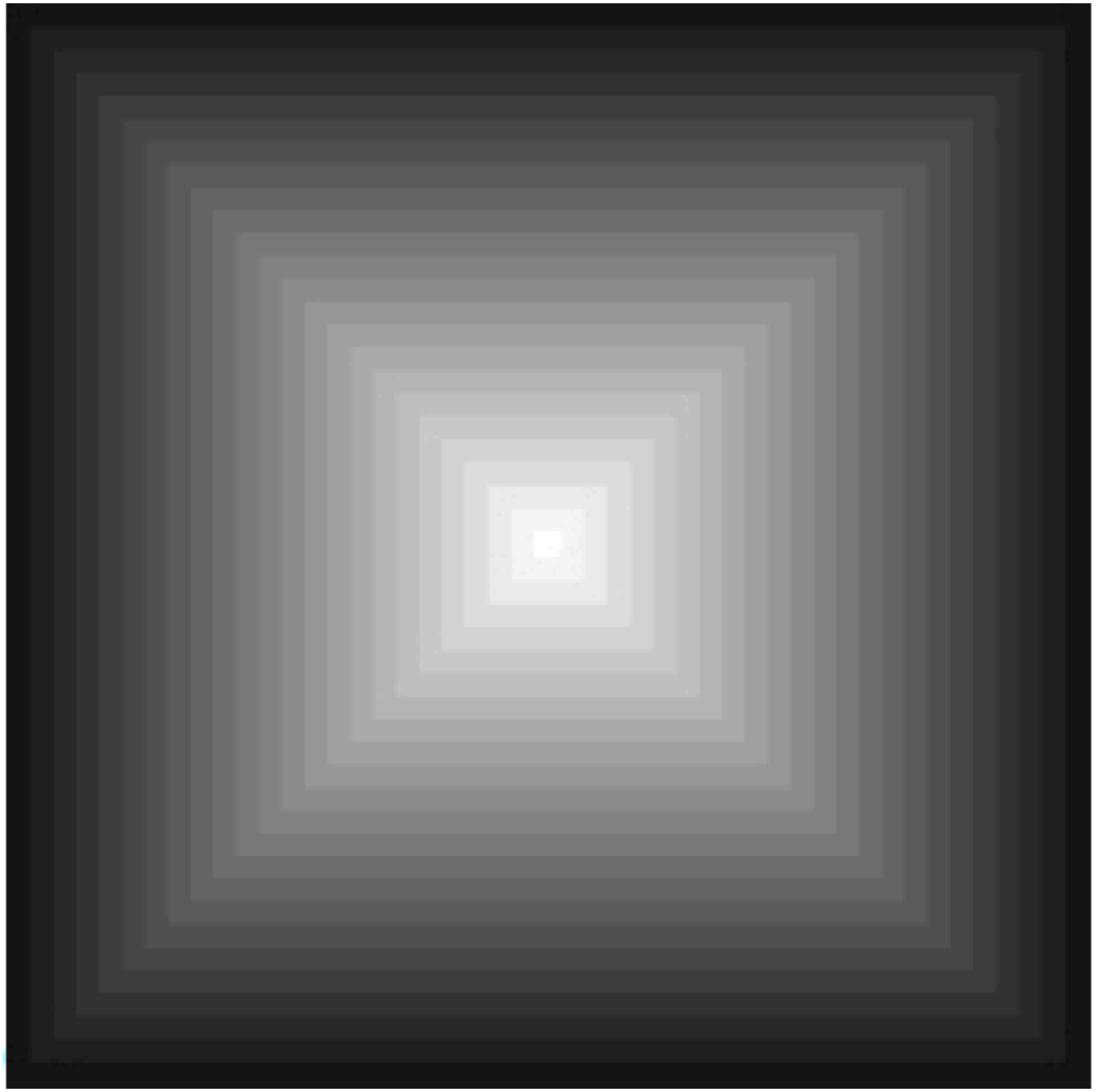
~ **6** cycles / degree of visual angle



Things that are very close
and/or large are hard to see

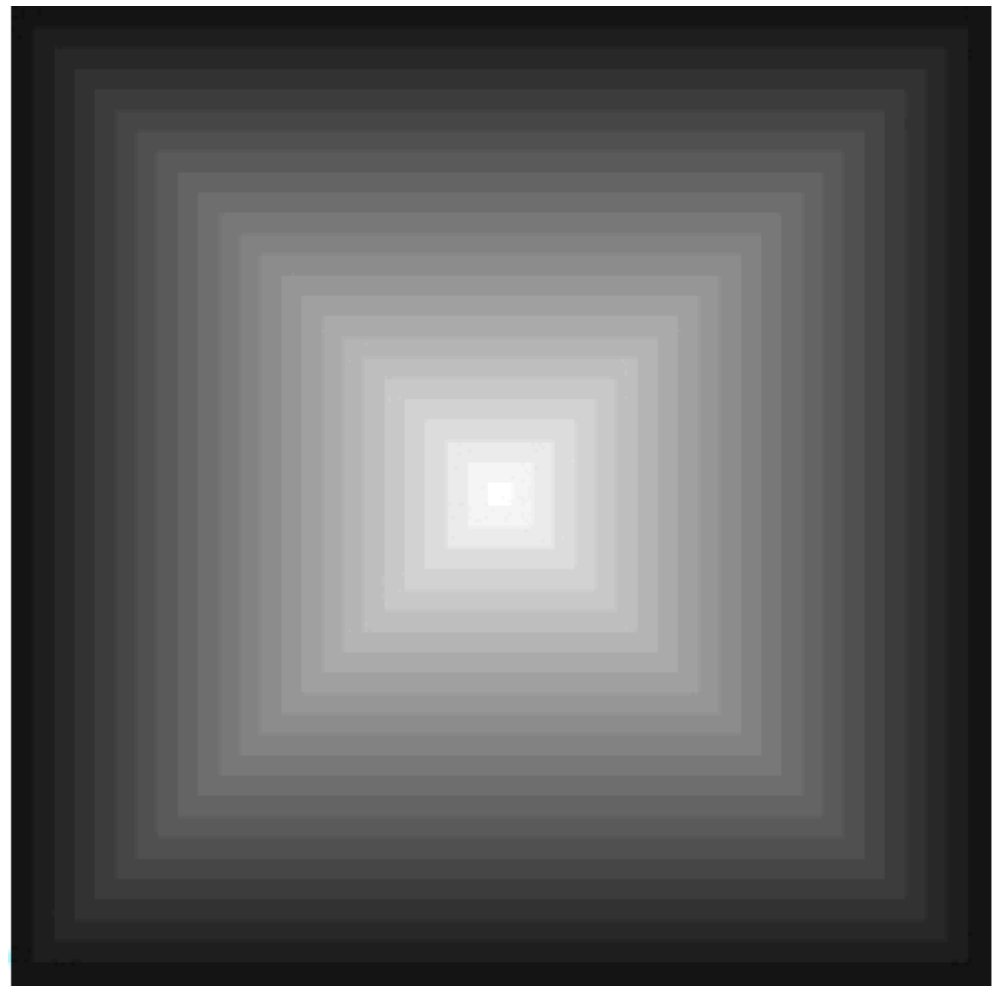
Things far away
are hard to see


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

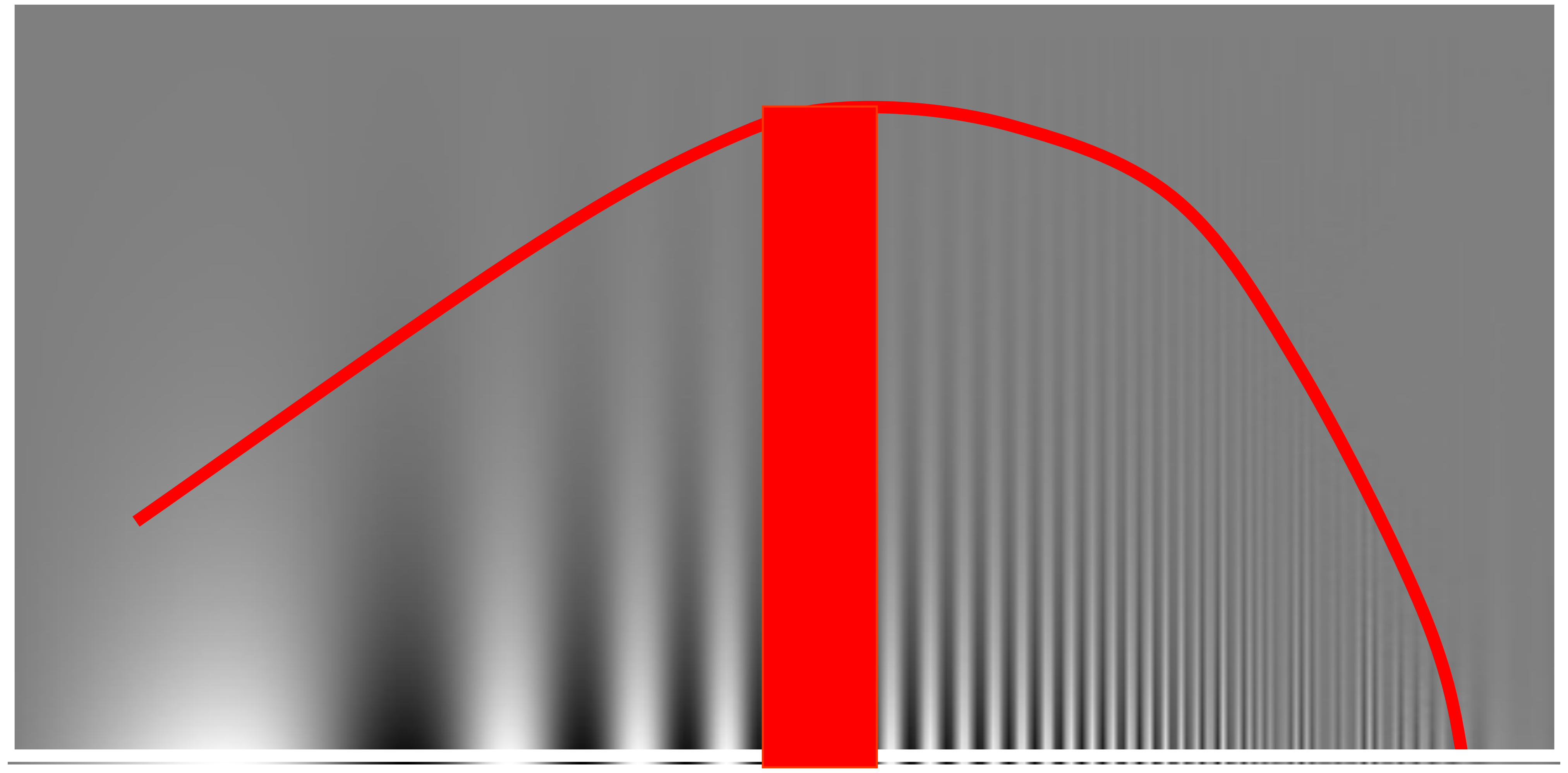


Vasarely visual illusion

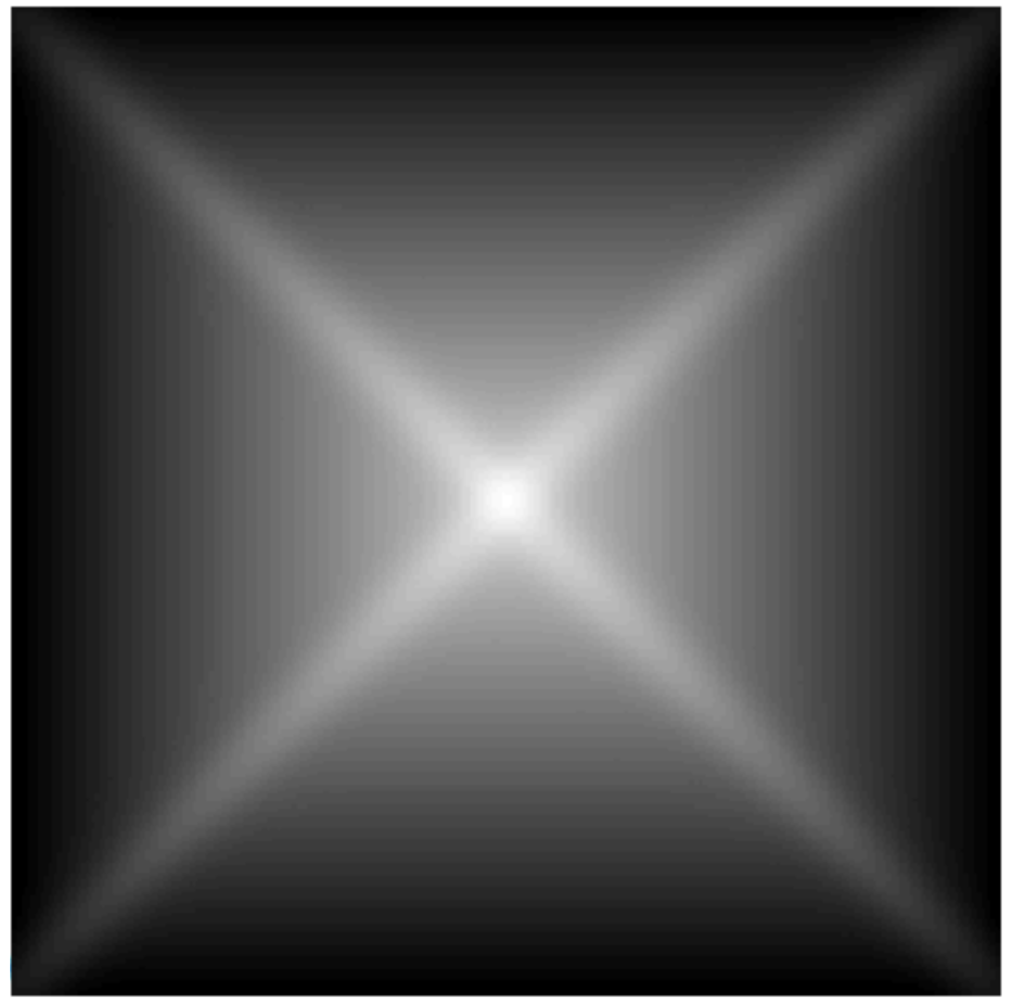
Input visual stimulus

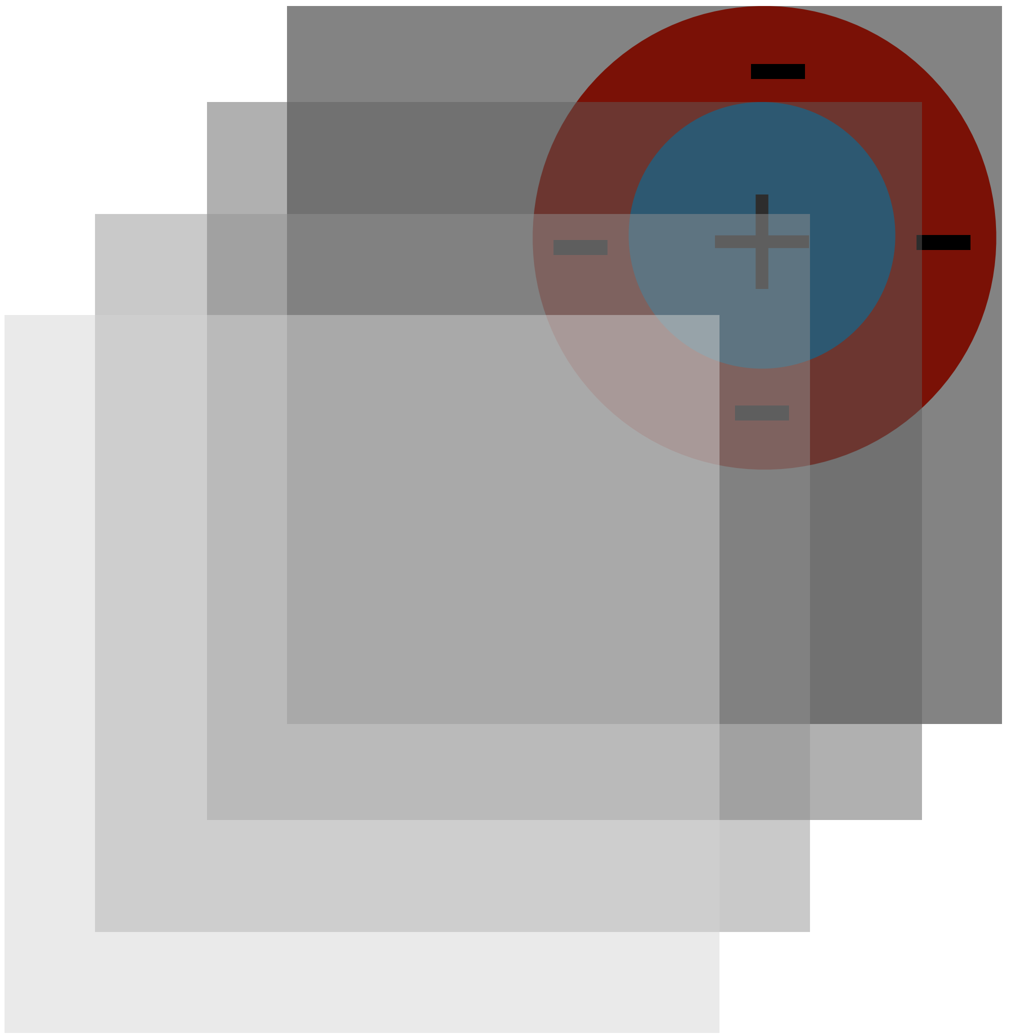


Frequency filtering of human visual system



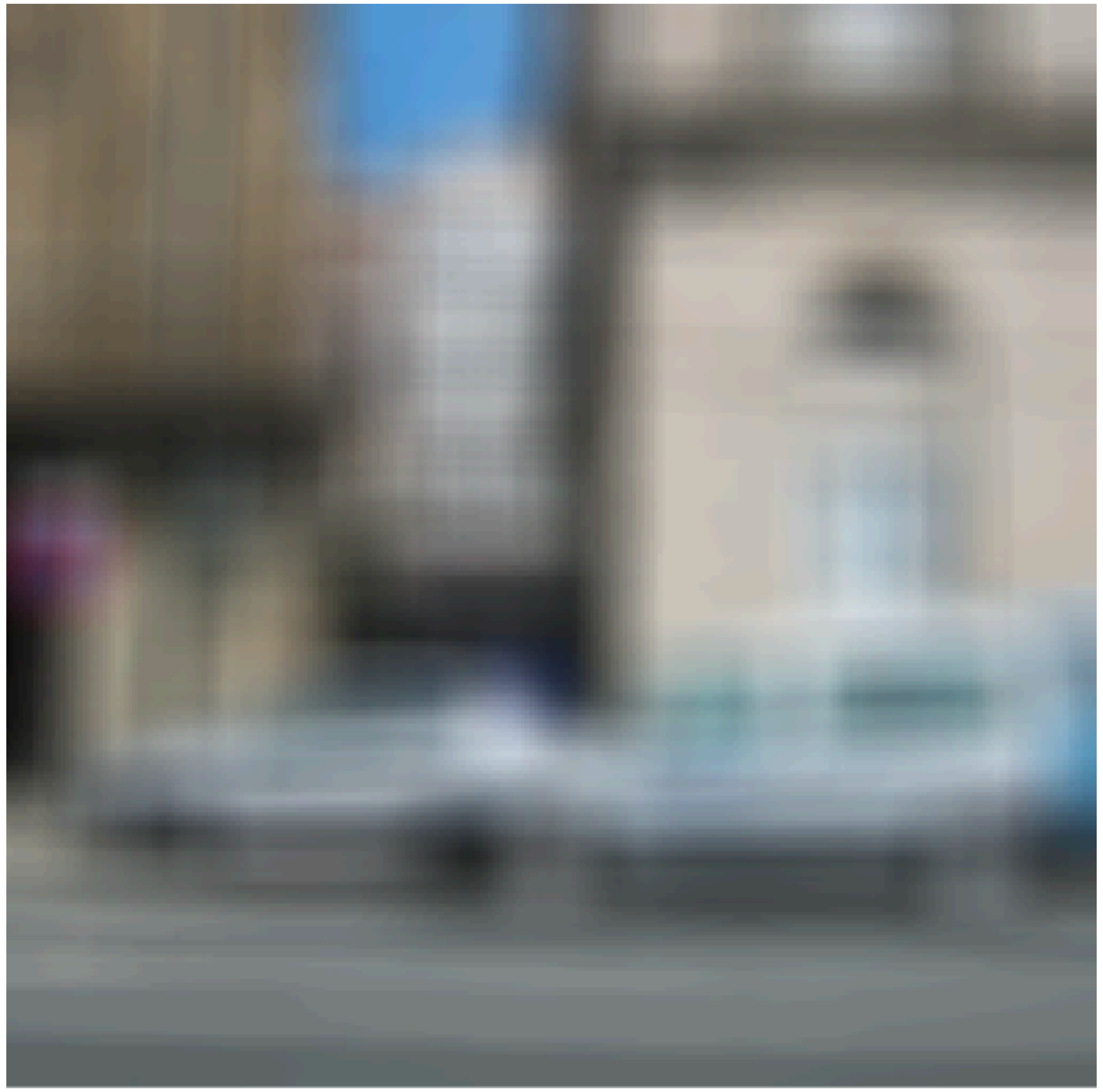
bandpass filtered output





Center-surround spatial filtering of human visual system is subtracting less positive intensity at the corners, giving a bright line there

Today: A collection of useful filters



Low-pass filters

High-pass filters

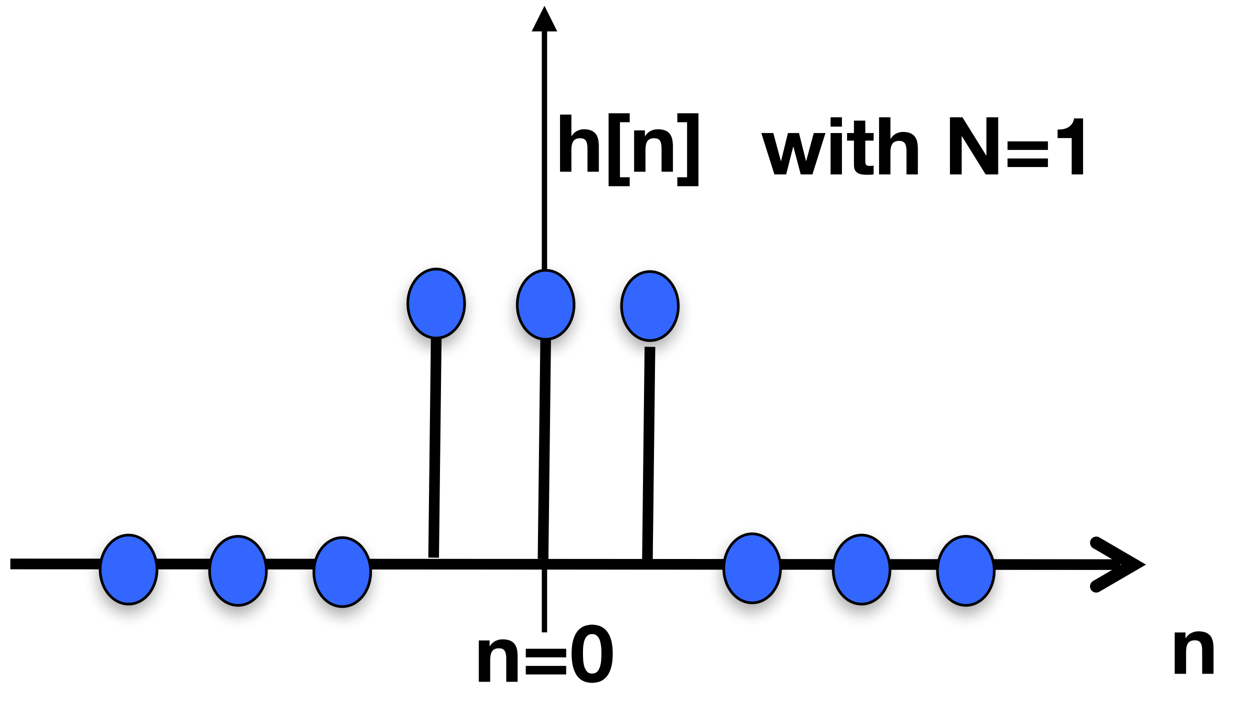
Low pass-filters

Box filter

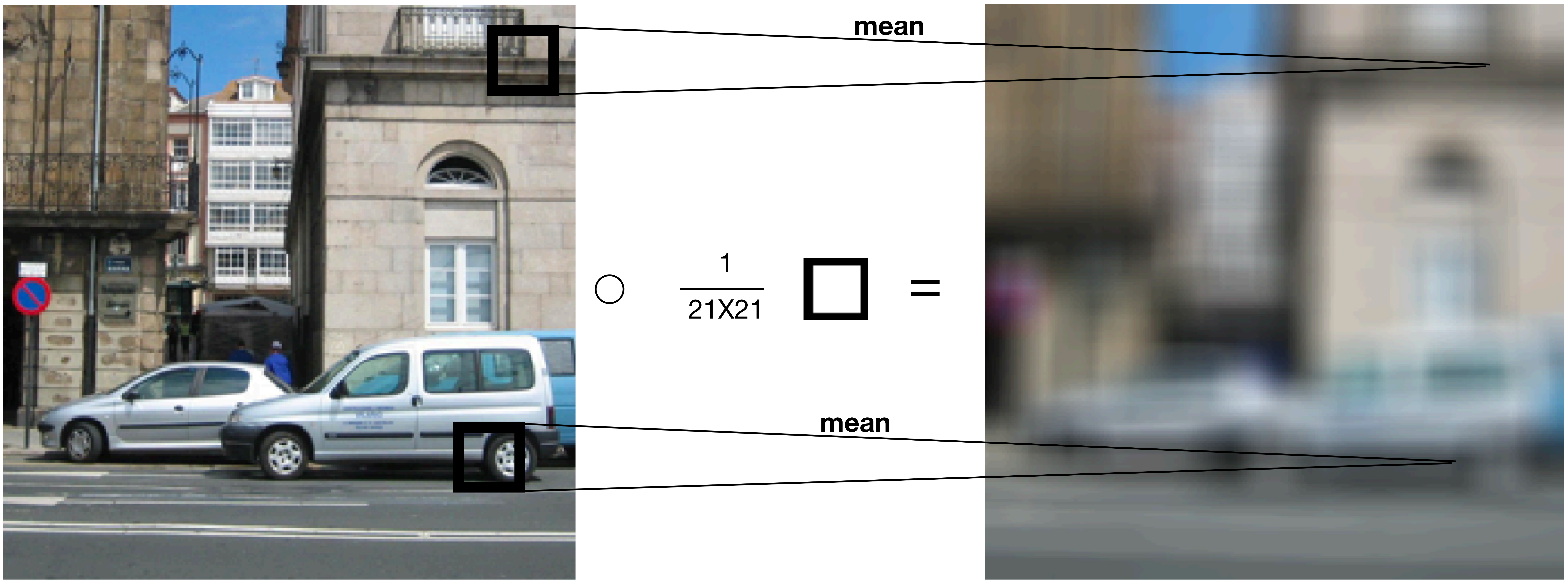
2N+1

1	1	…	1	
1	1		1	
1	1		1	2M+1
…				
1	1	1	1	

$$h_{N,M}[n,m] = \begin{cases} 1 & \text{if } -N \leq n \leq N \text{ and } -M \leq m \leq M \\ 0 & \text{otherwise} \end{cases}$$



Box filter



256X256 256X256

What does it do?

- Replaces each pixel with an average of its neighborhood
- Achieve smoothing effect (remove sharp features)

Box filter

The box filter is separable as it can be written as the convolution of two 1D kernels

$$h_{N,M}[n,m] = h_{N,0} \circ h_{0,M}$$

$$\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \bigcirc \begin{matrix} 1 & 1 \end{matrix} = \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$$

Box filter

256X256

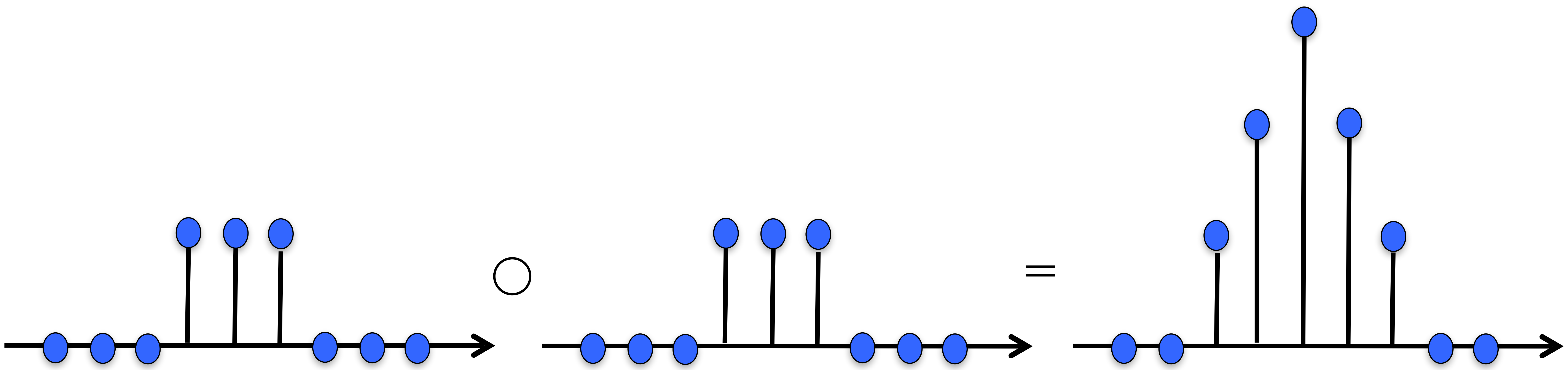
256X256

Requires N+N sums, instead of N*N

Box filter

If you convolve two boxes:

$$1\ \ 1\ \ 1 \ \circ\ 1\ \ 1\ \ 1 \ =\ 1\ \ 2\ \ 3\ \ 2\ \ 1$$

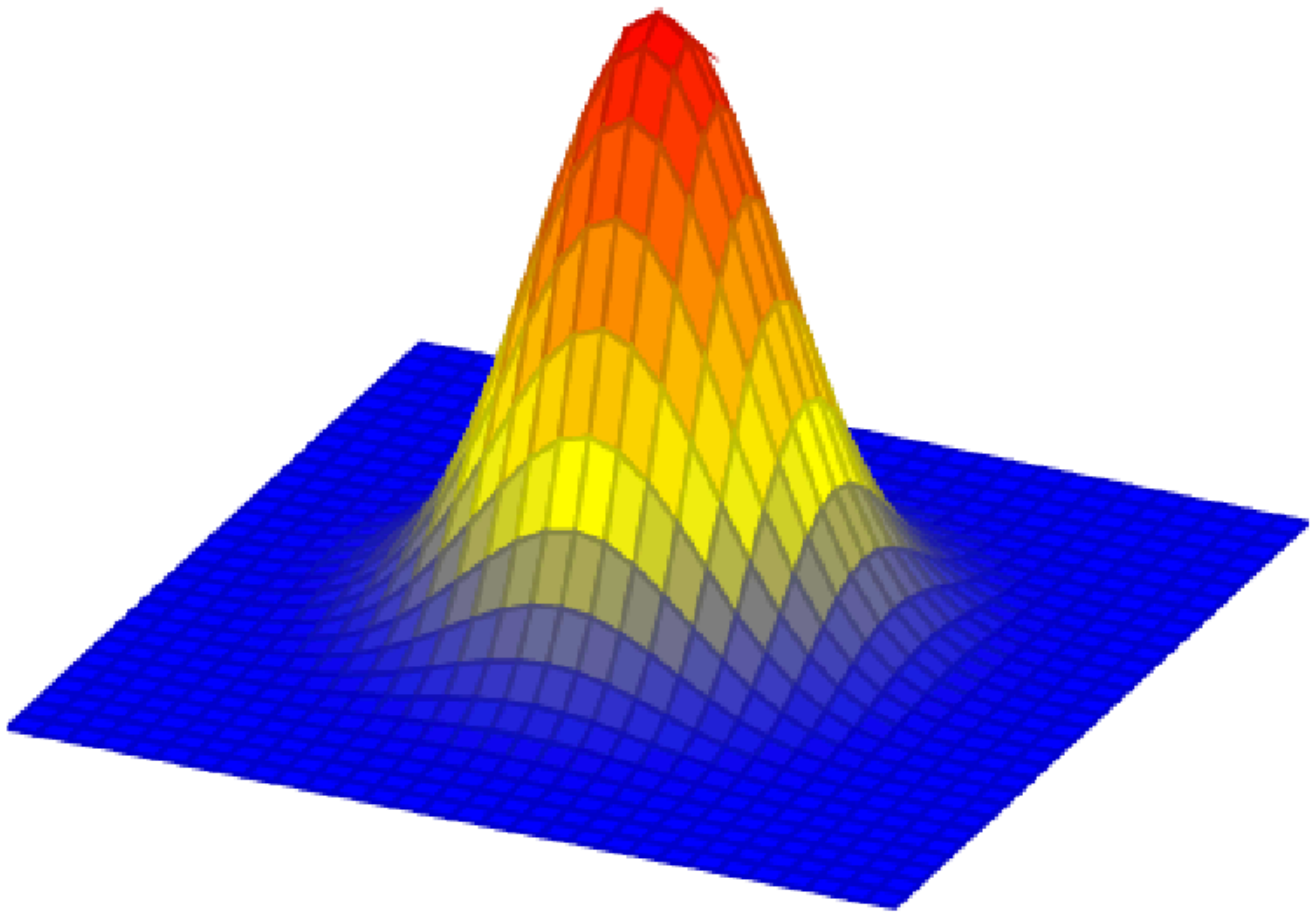


The convolution of two box filters is not another box filter.
It is a triangular filter.

Gaussian filter

In the continuous domain:

$$g(x, y; \sigma) = \frac{1}{2\pi\sigma^2} \exp -\frac{x^2 + y^2}{2\sigma^2}$$



Gaussian filter

$$g(x,y;\sigma) = \frac{1}{2\pi\sigma^2}\exp - \frac{x^2+y^2}{2\sigma^2}$$

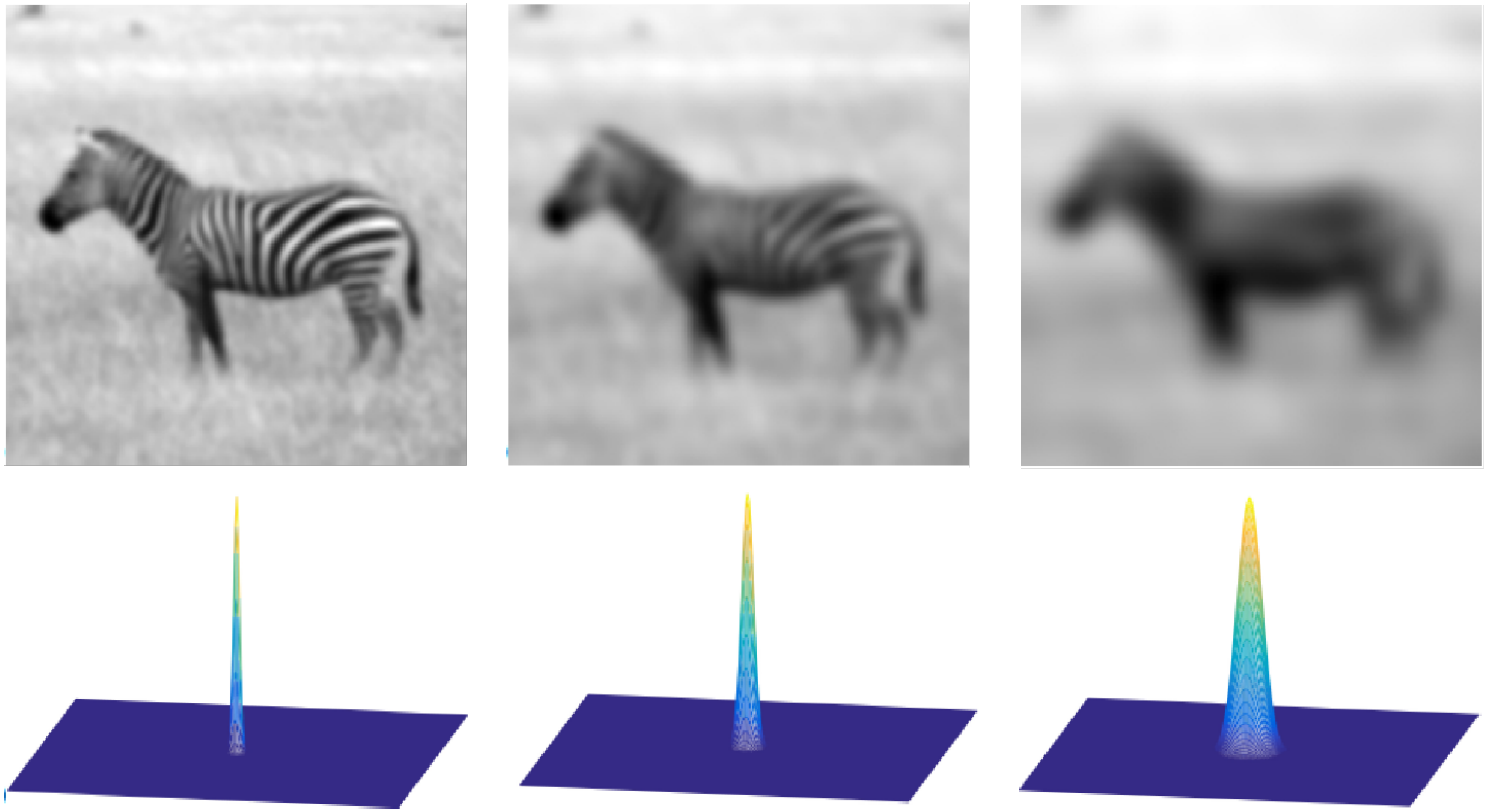
Discretization of the Gaussian:

At 3σ the amplitude of the Gaussian is around 1% of its central value

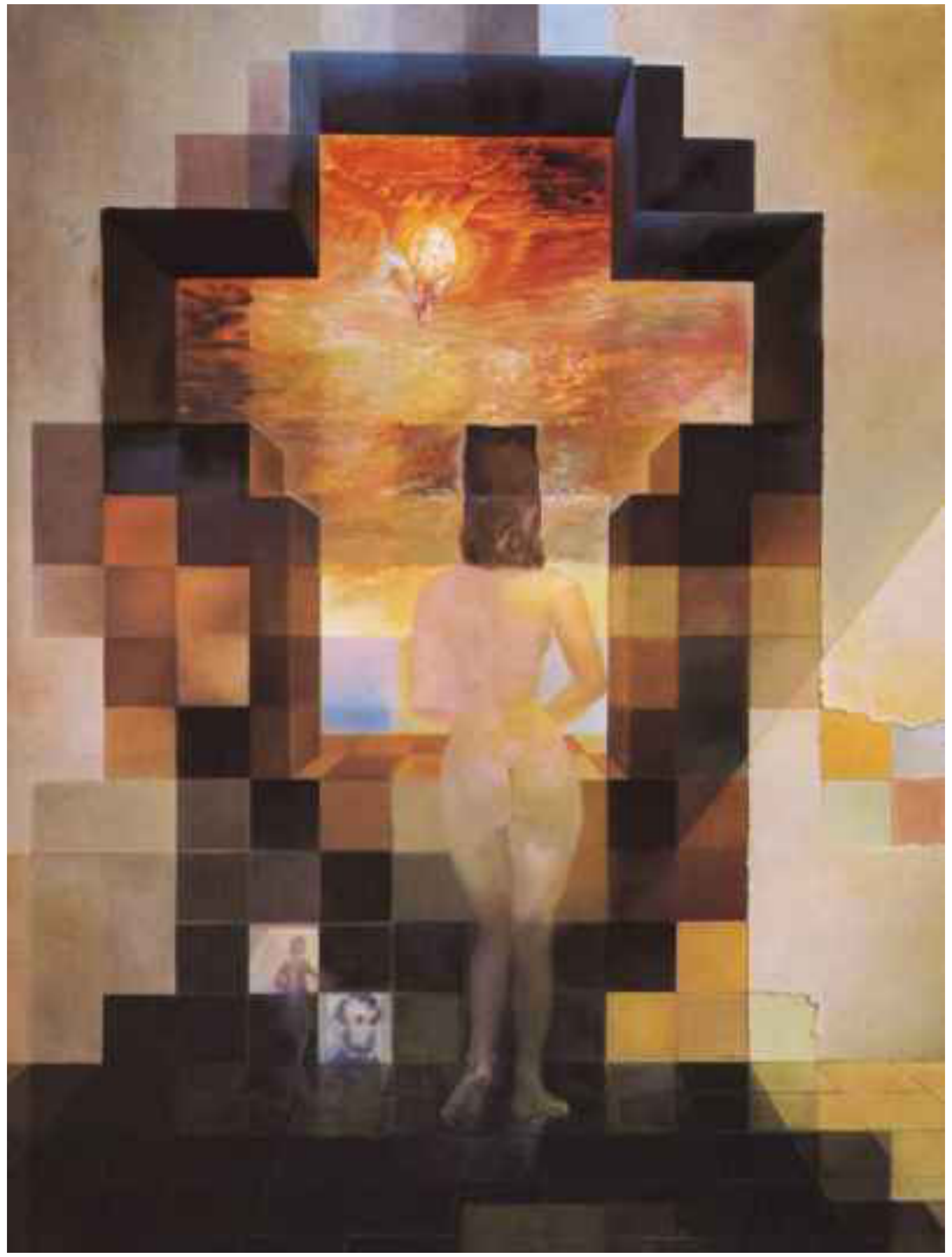
$$g[m,n;\sigma] = \exp - \frac{m^2+n^2}{2\sigma^2}$$

Scale

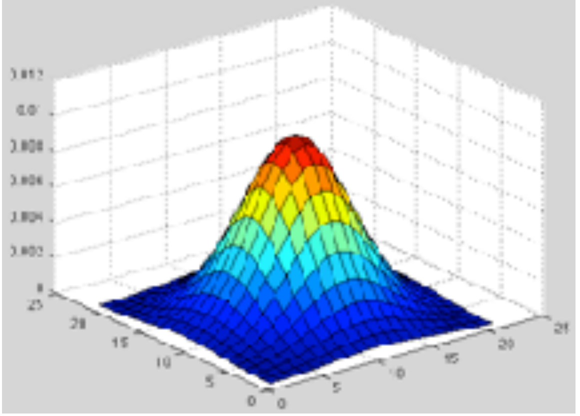
$$g[m,n;\sigma] = \exp -\frac{m^2+n^2}{2\sigma^2}$$



Gaussian filter for low-pass filtering



Dali



Properties of the Gaussian filter

$$g(x,y;\sigma) = \frac{1}{2\pi\sigma^2} \exp -\frac{x^2+y^2}{2\sigma^2}$$

- The n-dimensional Gaussian is the only completely circularly symmetric operator that is separable.

- The (continuous) Fourier transform of a Gaussian is another Gaussian

$$G(u,v;\sigma) = \exp -2\pi^2(u^2+v^2)\sigma^2$$

Properties of the Gaussian filter

$$g(x, y; \sigma) = \frac{1}{2\pi\sigma^2} \exp - \frac{x^2 + y^2}{2\sigma^2}$$

- The convolution of two n-dimensional Gaussians is an n-dimensional Gaussian.

$$g(x, y; \sigma_1) \circ g(x, y; \sigma_2) = g(x, y; \sigma_3)$$

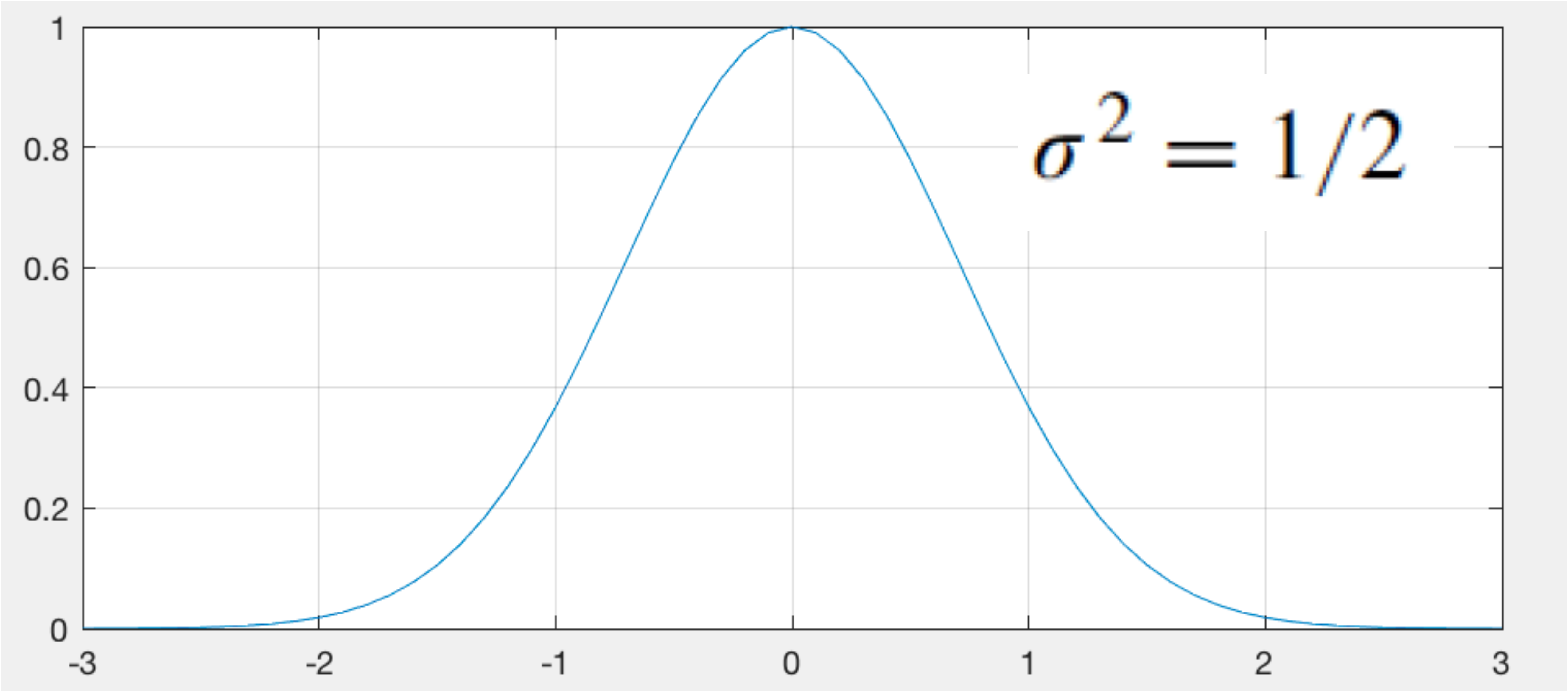
where the variance of the result is the sum

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

(it is easy to prove this using the FT of the Gaussian)

Discretization of the Gaussian

There are very efficient approximations to the Gaussian filter for certain values of σ with nicer properties than when working with discretized gaussians.



$$g_5[n] = [0.0183,\ 0.3679,\ 1.0000,\ 0.3679,\ 0.0183]$$

Binomial filter

Binomial coefficients provide a compact approximation of the gaussian coefficients using only integers.

The simplest blur filter (low pass) is

$$[1 \;\; 1]$$

Binomial filters in the family of filters obtained as successive convolutions of [1 1]

Binomial filter

$$b_1 = [1 \;\; 1]$$

$$b_2 = [1\ 1] \circ [1\ 1] = [1\ 2\ 1]$$

$$b_3 = [1\ 1] \circ [1\ 1] \circ [1\ 1] = [1\ 3\ 3\ 1]$$

Binomial filter

b_1	1	1								$\sigma_1^2 = 1/4$
b_2	1	2	1							$\sigma_2^2 = 1/2$
b_3	1	3	3	1						$\sigma_3^2 = 3/4$
b_4	1	4	6	4	1					$\sigma_4^2 = 1$
b_5	1	5	10	10	5	1				$\sigma_5^2 = 5/4$
b_6	1	6	15	20	15	6	1			$\sigma_6^2 = 3/2$
b_7	1	7	21	35	35	21	7	1		$\sigma_7^2 = 7/4$
b_8	1	8	28	56	70	56	28	8	1	$\sigma_8^2 = 2$

Properties of binomial filters

- Sum of the values is 2^n
- The variance of b_n is $\sigma^2 = n/4$
- The convolution of two binomial filters is also a binomial filter

$$b_n \circ b_m = b_{n+m}$$

With a variance:

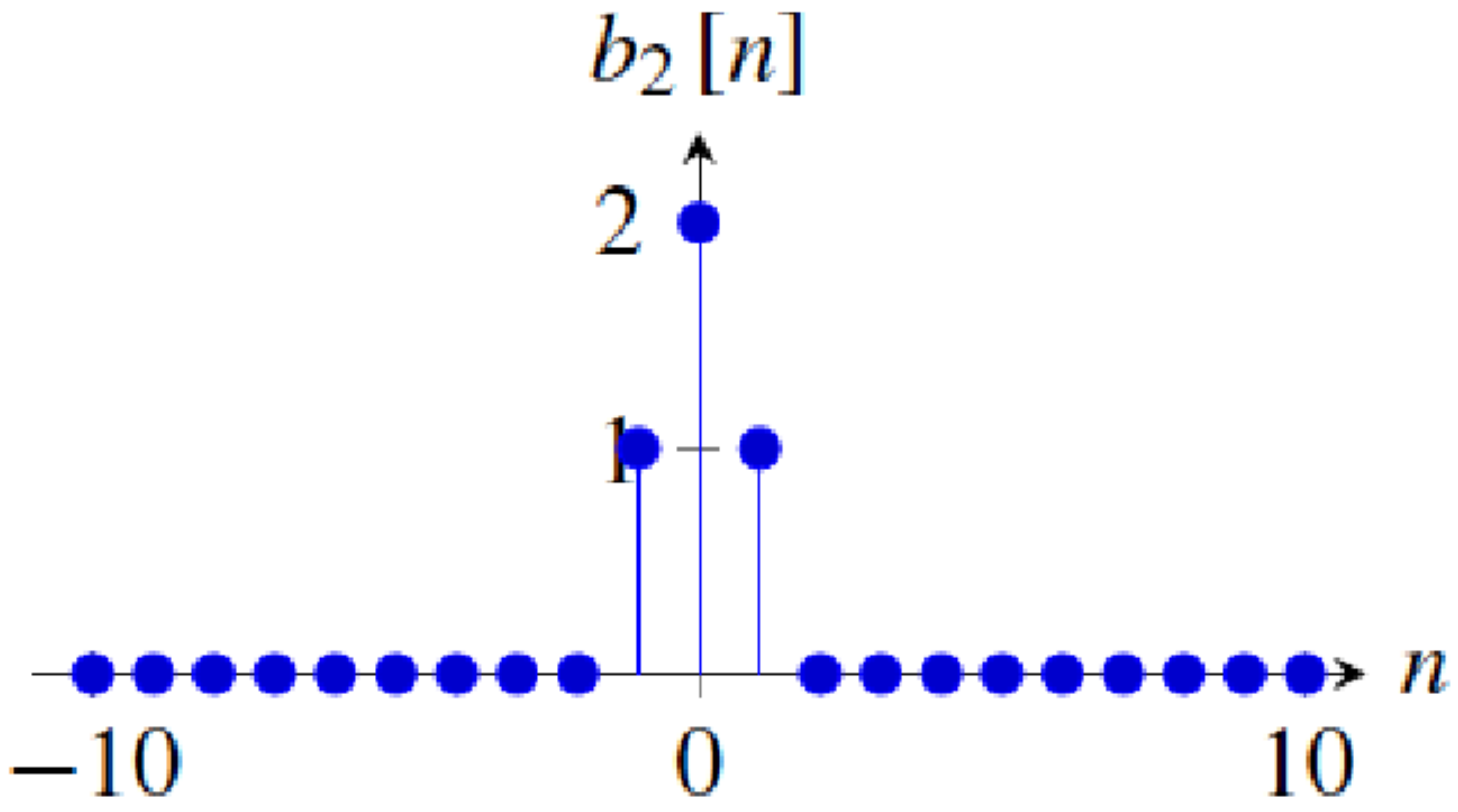
$$\sigma_n^2 + \sigma_m^2 = \sigma_{n+m}^2$$

These properties are analogous to the gaussian property in the continuous domain (but the binomial filter is different than a discretization of a gaussian)

B2[n]

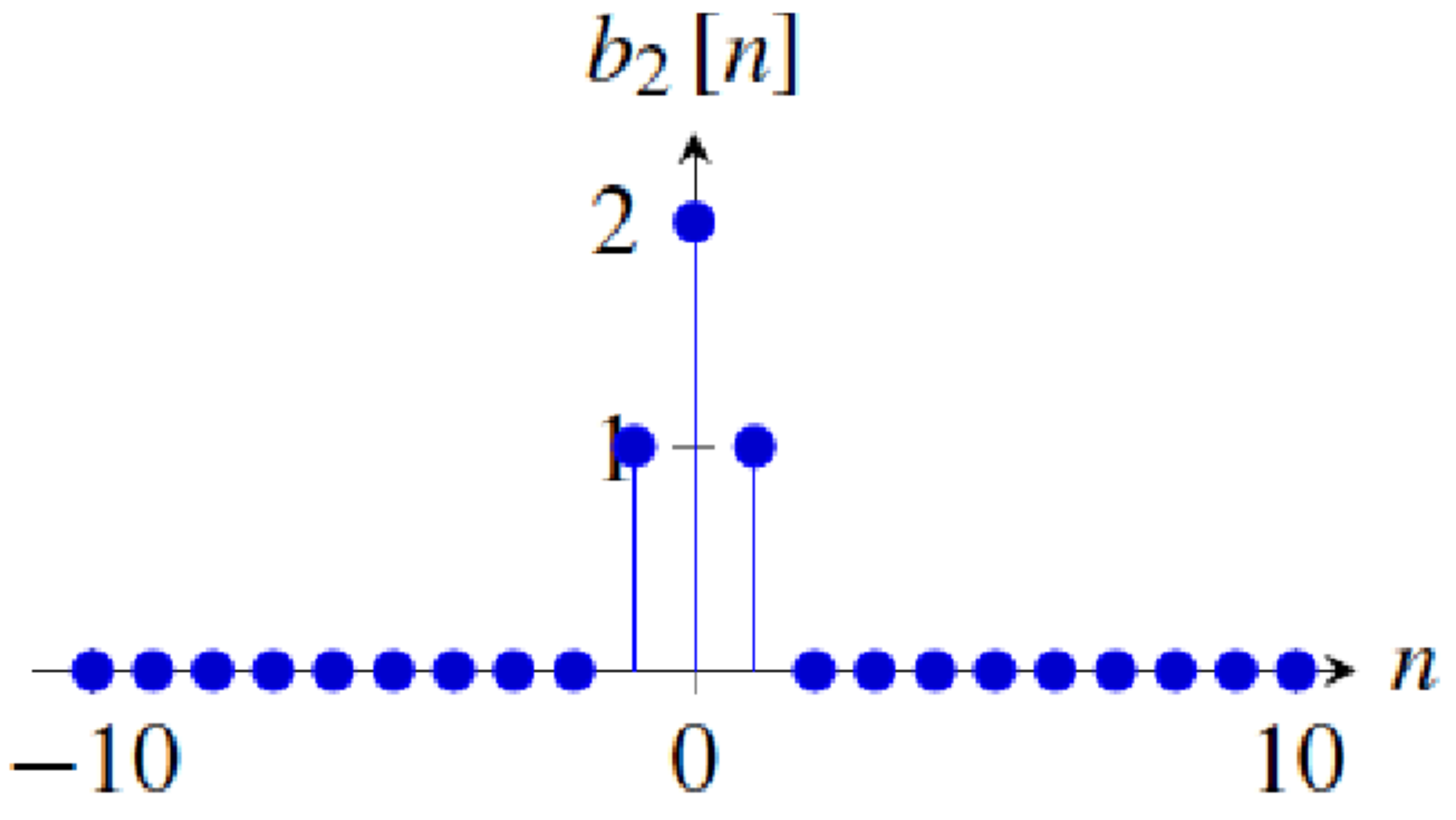
The simplest approximation to the Gaussian filter is the 3-tap kernel:

$$b_2 = [1, 2, 1]$$

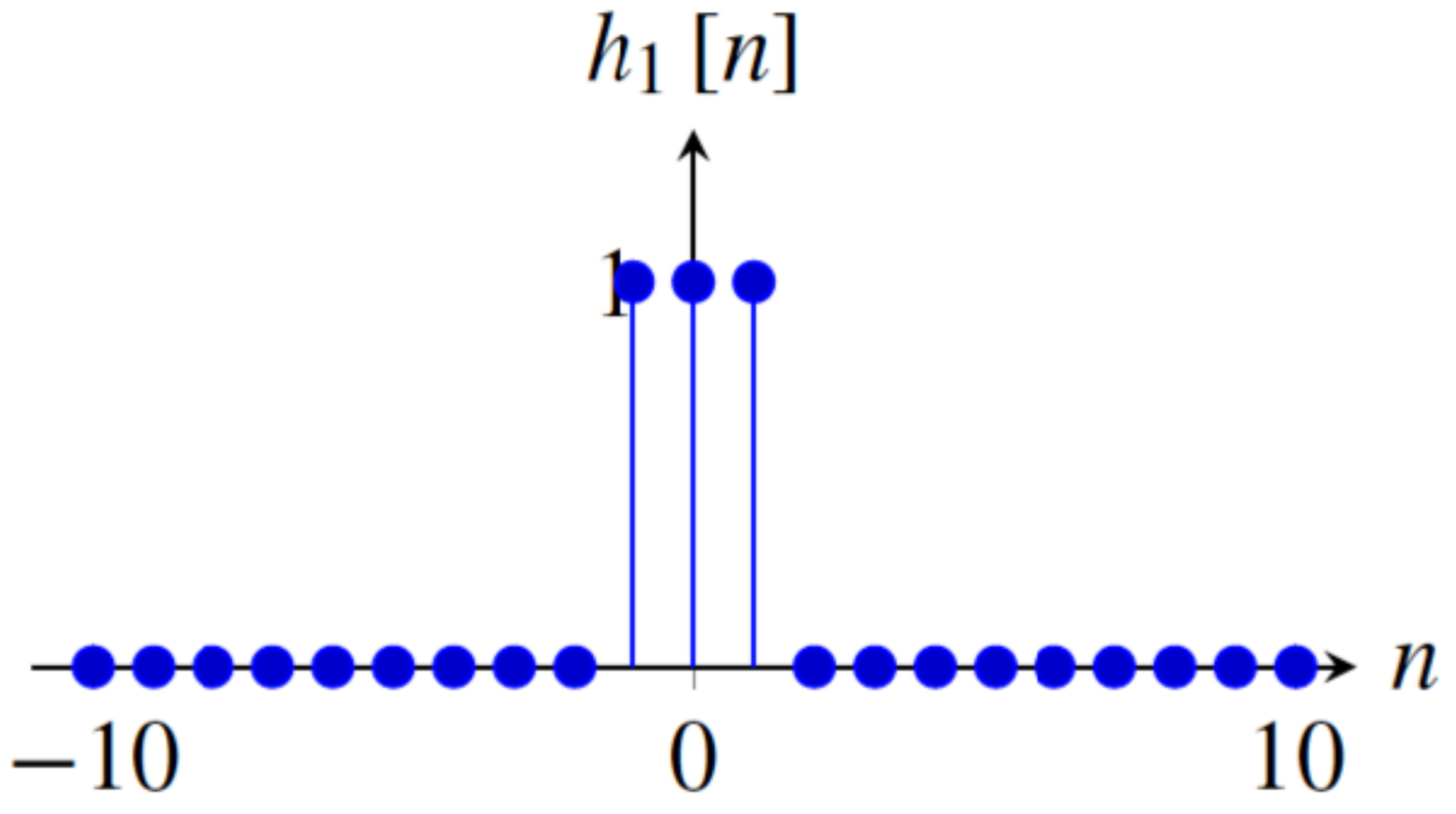


B2[n] versus the 3-tap box filter

[1 2 1]



[1 1 1]



Which one is better?

B2[n]

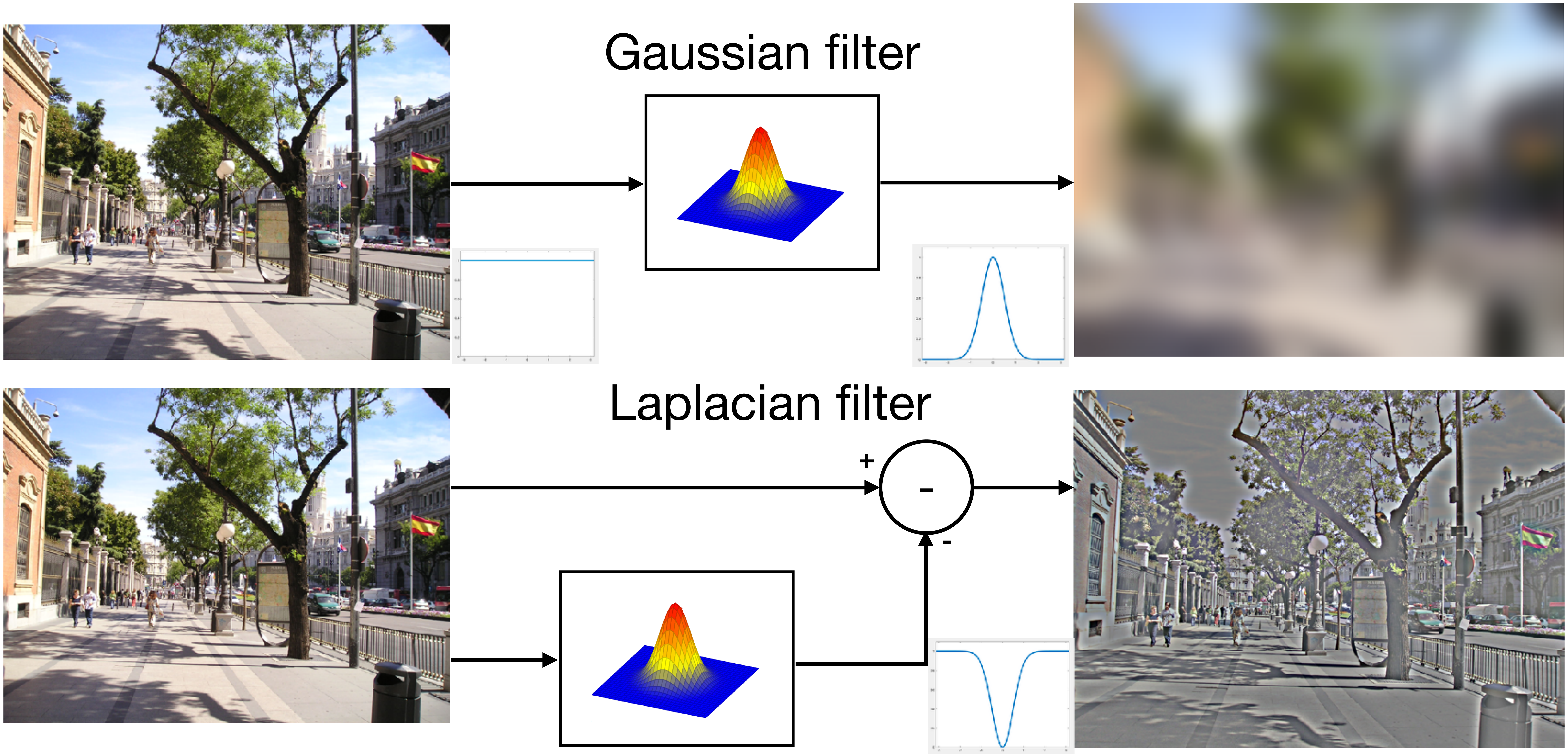
$$[1, 1, 1] \circ [\ldots, 1, -1, 1, -1, 1, -1, \ldots] = [\ldots, -1, 1, -1, 1, -1, 1, \ldots]$$

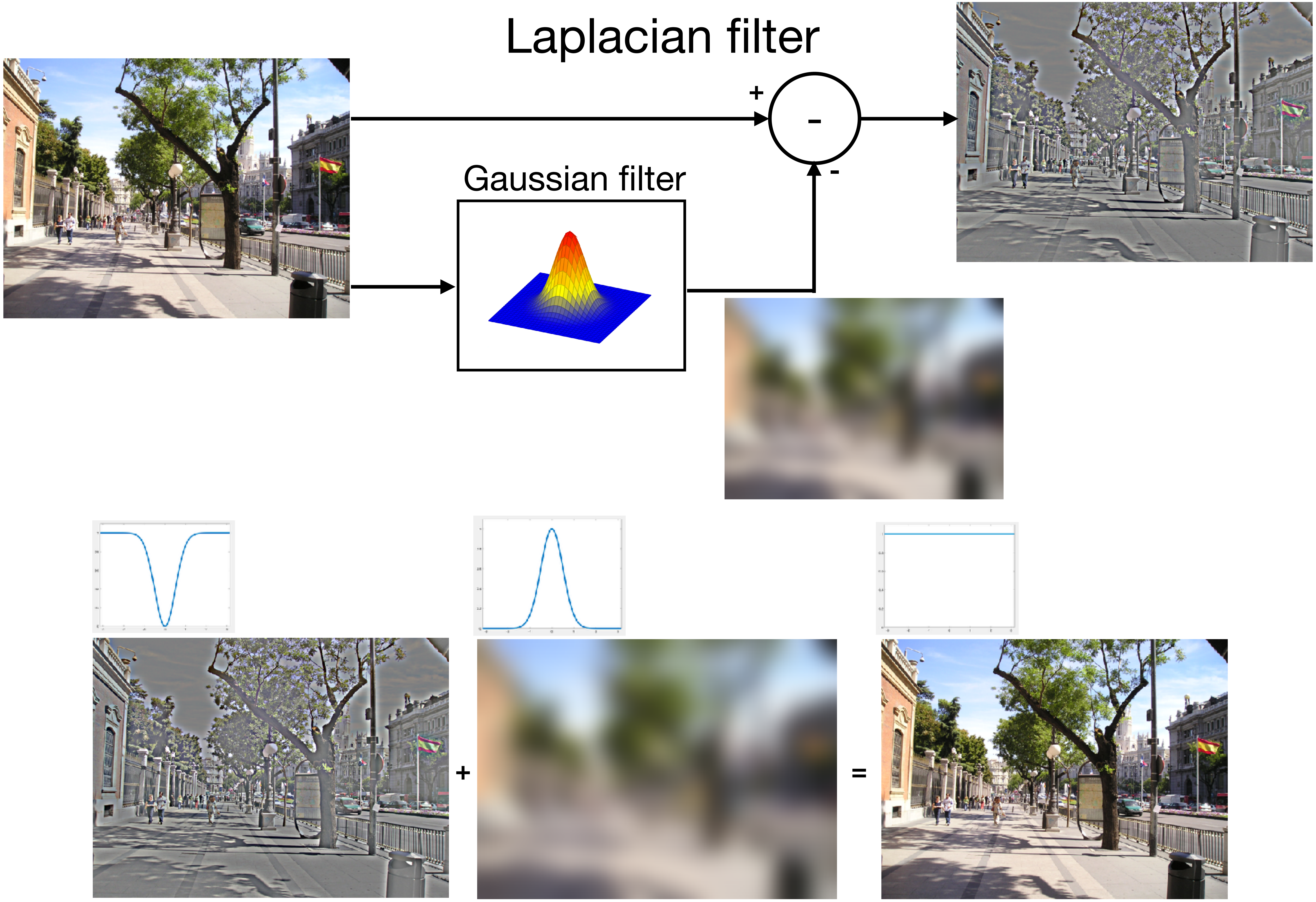
$$[1, 2, 1] \circ [\ldots, 1, -1, 1, -1, 1, -1, \ldots] = [\ldots, 0, 0, 0, 0, 0, 0, \ldots]$$

B2[n]

$$b_{2,2} = b_{2,0} \circ b_{0,2} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

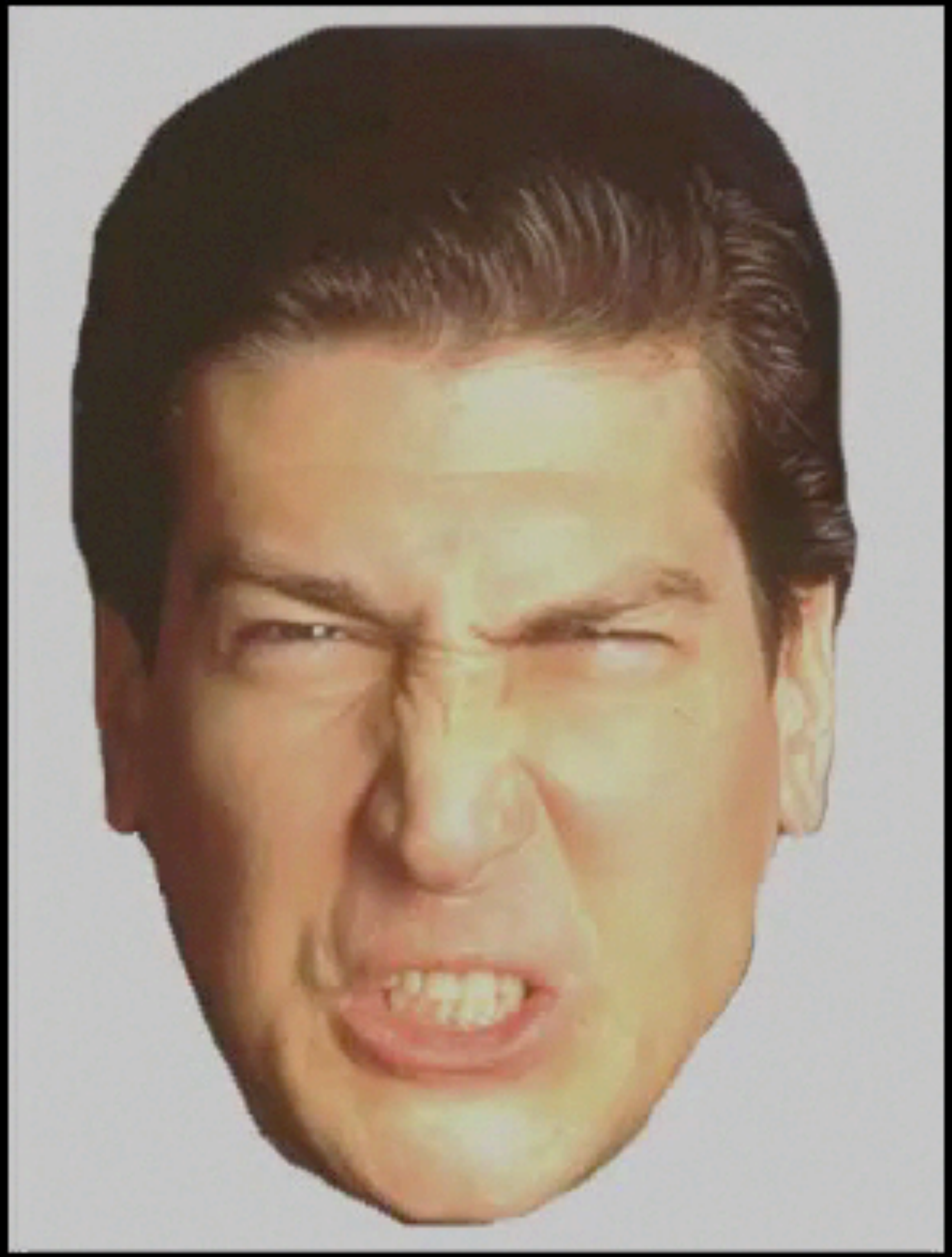
What about the opposite of blurring?



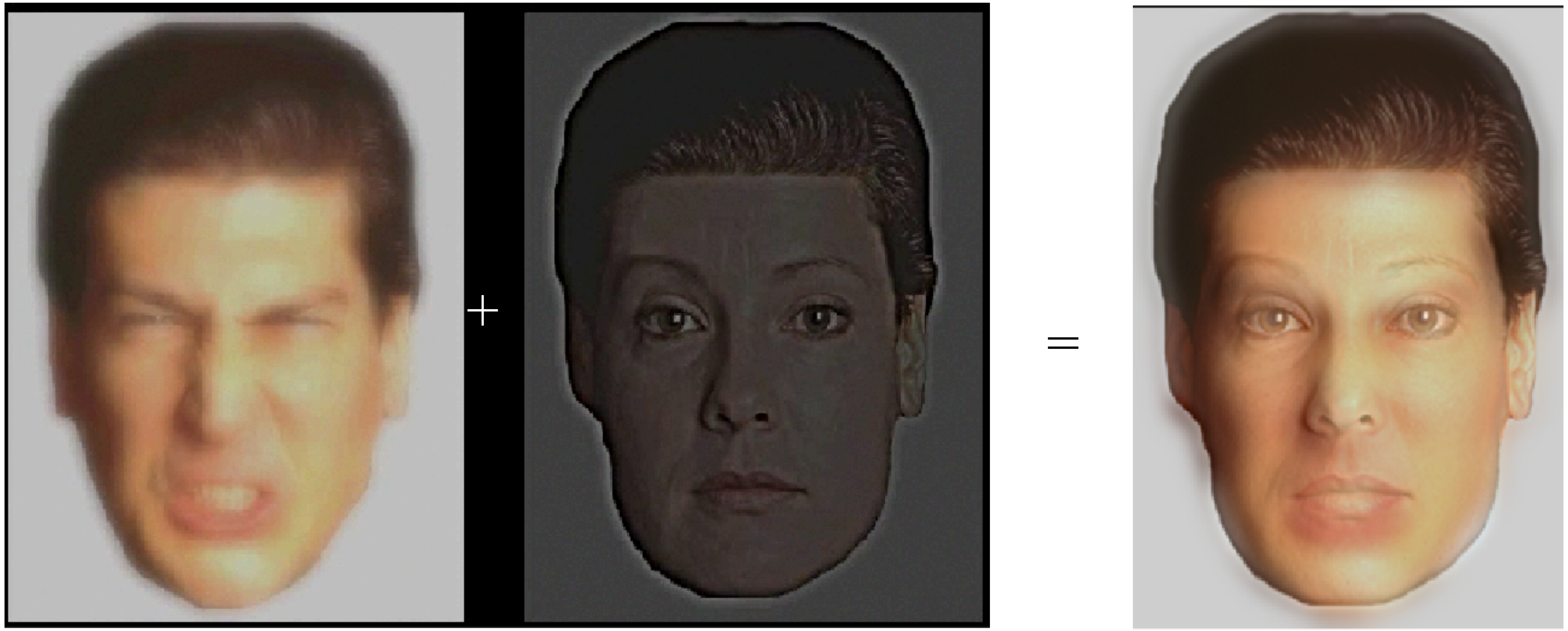

Laplacian filter
+
-
-
Gaussian filter
+
=

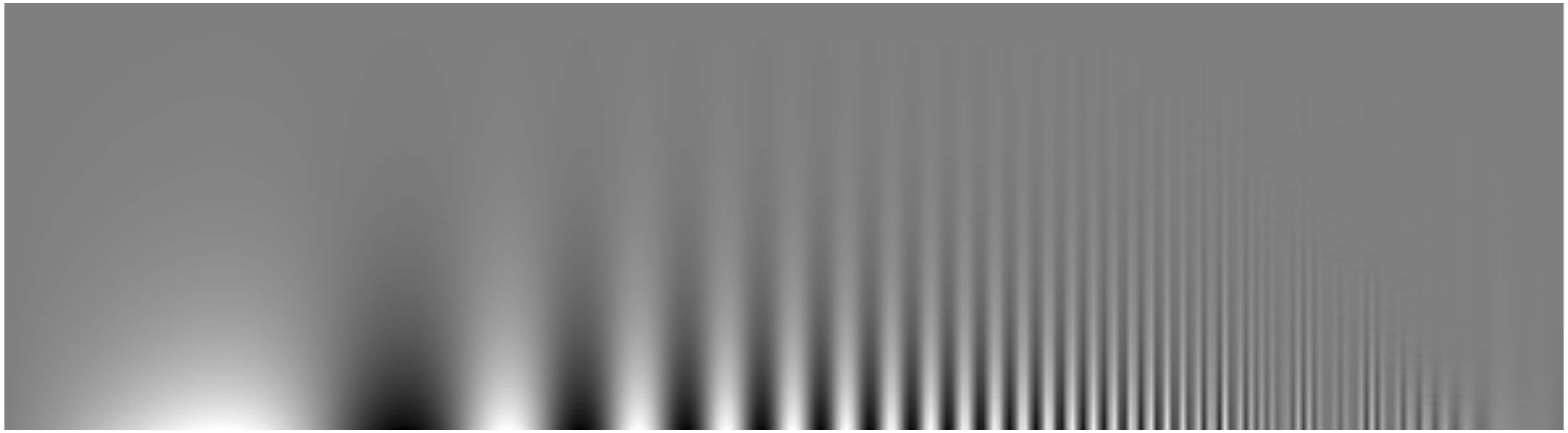
Hybrid Images

Oliva & Schyns

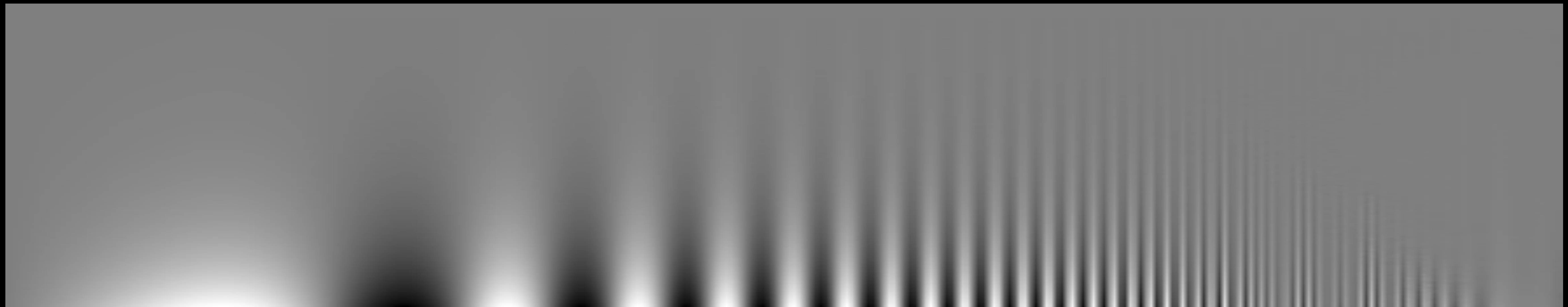


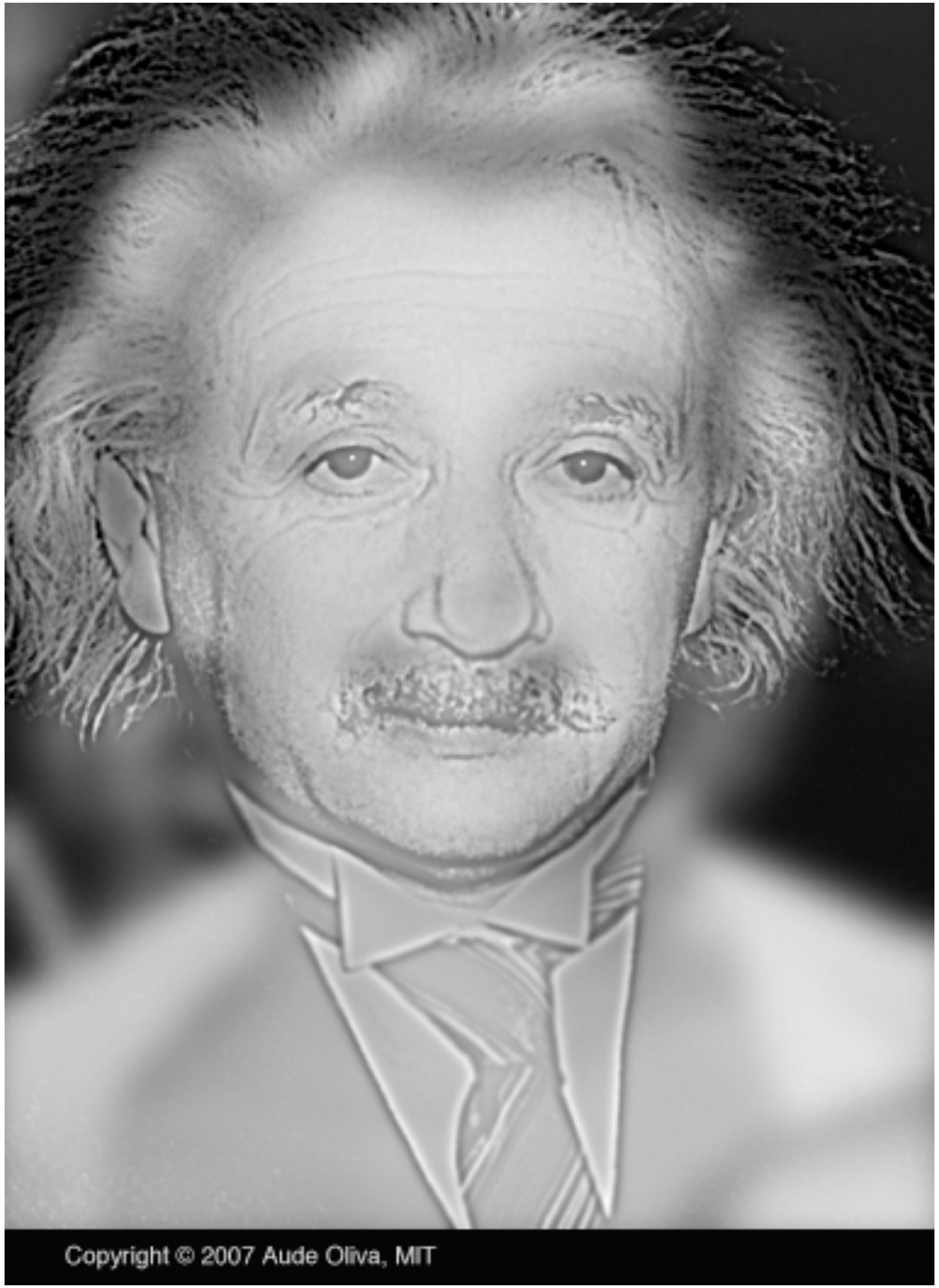
Hybrid Images

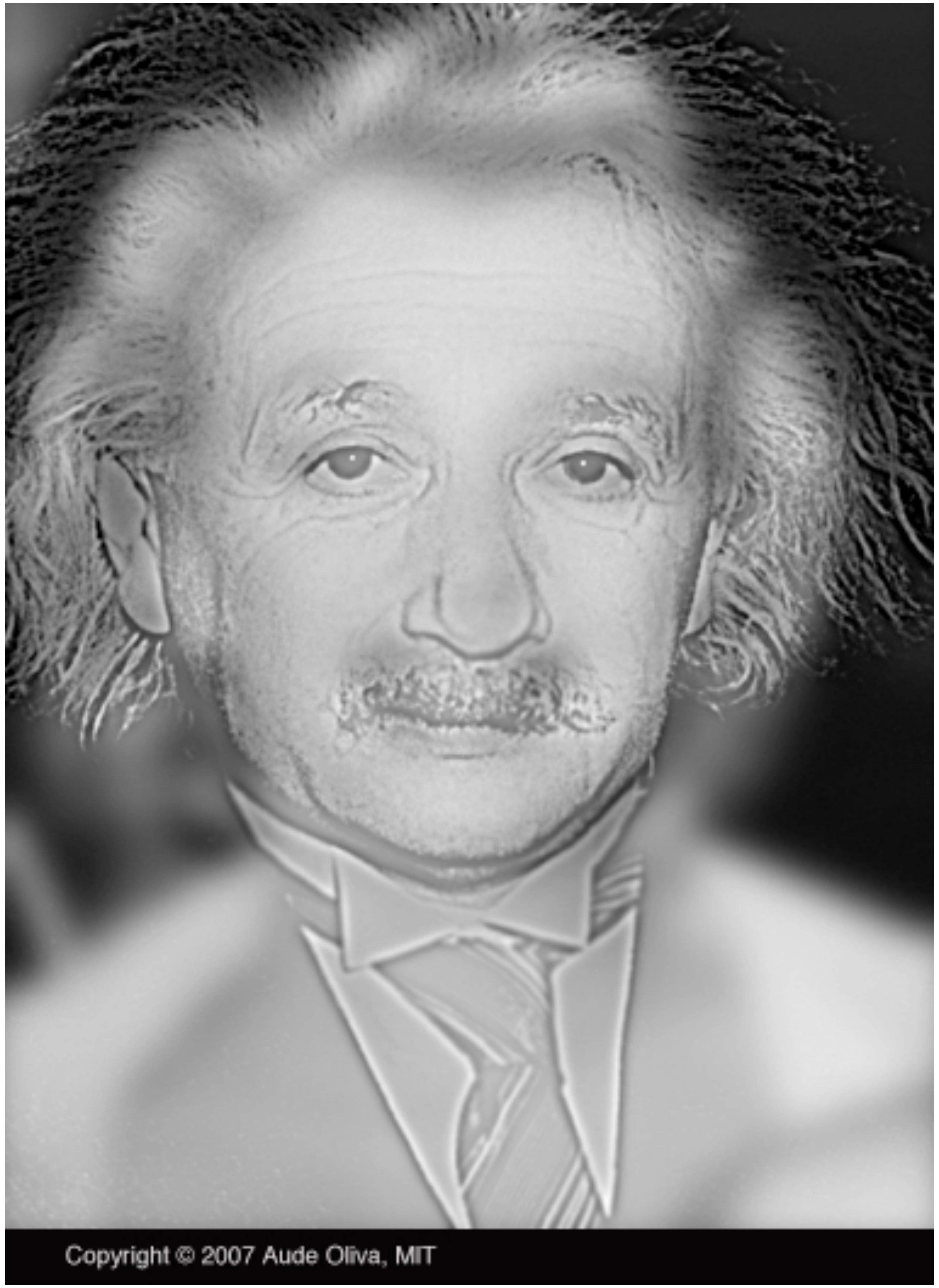


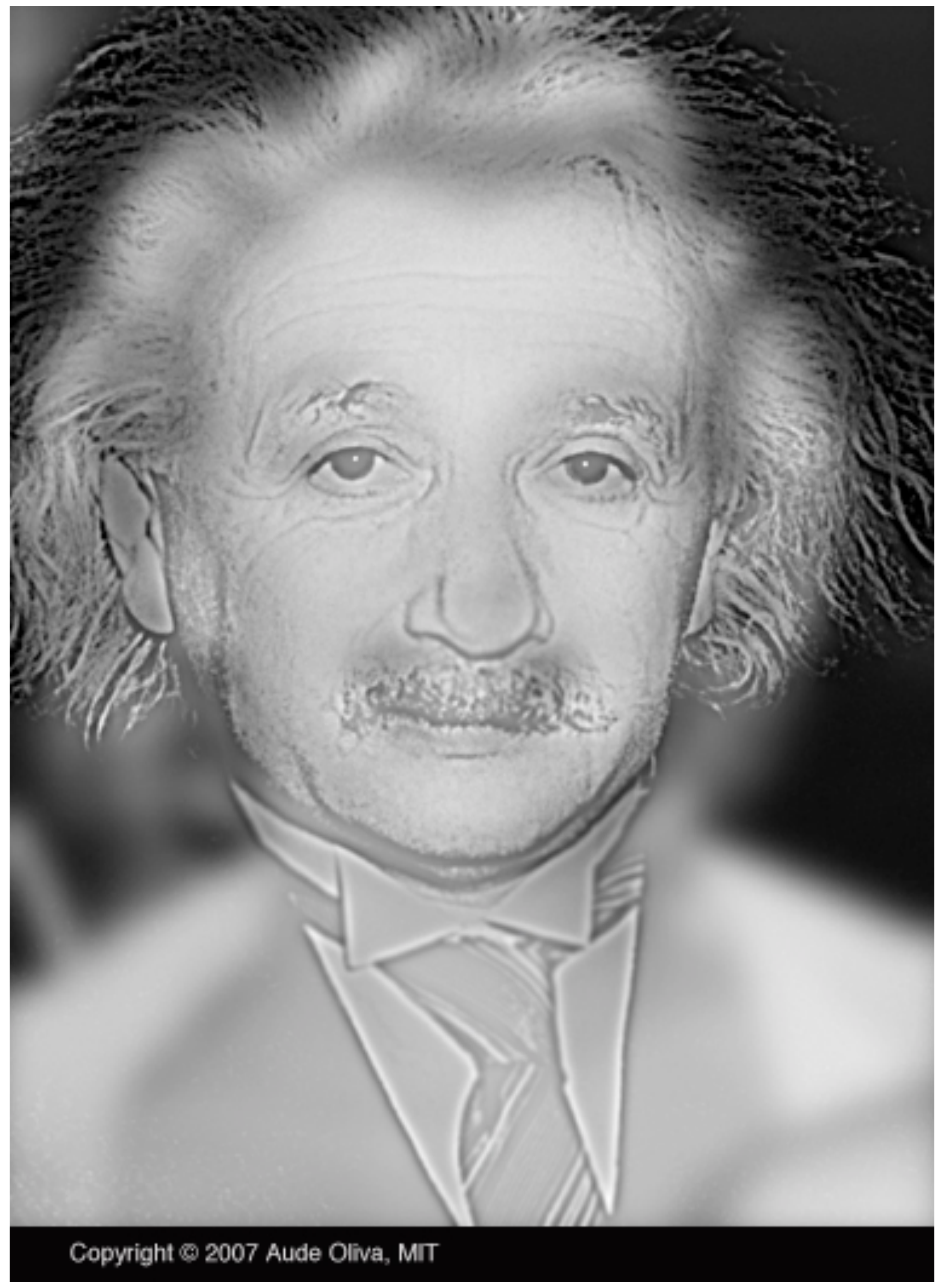


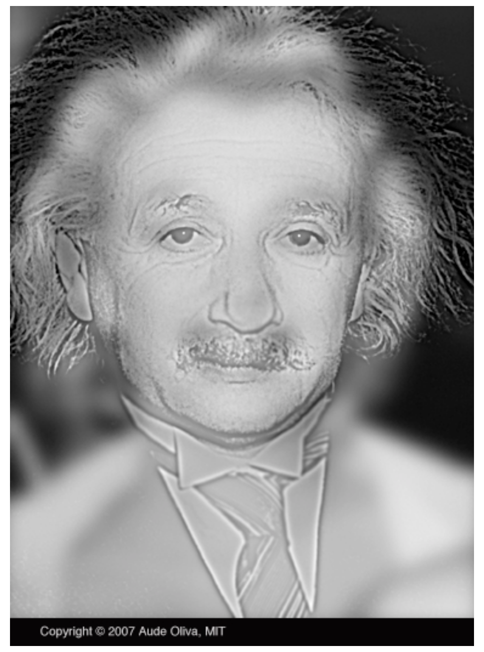
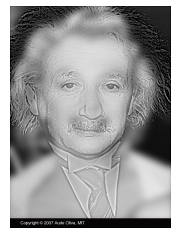
Hybrid Images




Copyright © 2007 Aude Oliva, MIT





http://cvcl.mit.edu/hybrid_gallery/gallery.html

High pass-filters

Finding edges in the image

Image gradient:

$$\nabla \mathbf{I} = \left(\frac{\partial \mathbf{I}}{\partial x}, \frac{\partial \mathbf{I}}{\partial y} \right)$$

Approximation image derivative:

$$\frac{\partial \mathbf{I}}{\partial x} \simeq \mathbf{I}(x, y) - \mathbf{I}(x-1, y)$$

Edge strength $\quad E(x, y) = |\nabla \mathbf{I}(x, y)|$

Edge orientation: $\quad \theta(x, y) = \angle \nabla \mathbf{I} = \arctan \frac{\partial \mathbf{I}/\partial y}{\partial \mathbf{I}/\partial x}$

Edge normal: $\quad \mathbf{n} = \frac{\nabla \mathbf{I}}{|\nabla \mathbf{I}|}$

[-1 1]

$$\frac{\partial \mathbf{I}}{\partial x} \simeq \mathbf{I}(x,y) - \mathbf{I}(x-1,y)$$

$\otimes$ [-1, 1] =

h[m,n]

f[m,n]

g[m,n]

$[-1\ 1]^T$

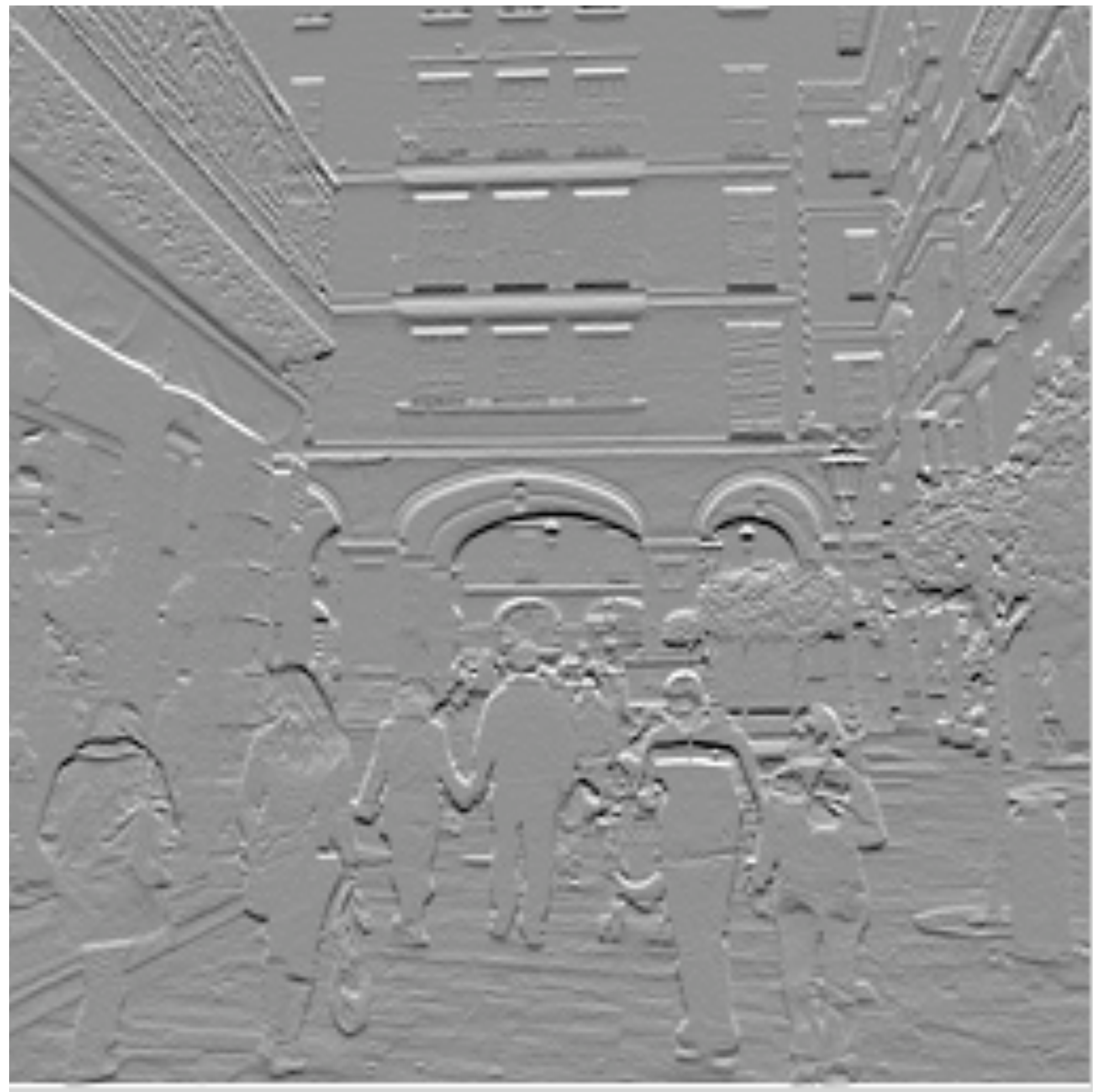
g[m,n]

$\otimes$

$[-1, 1]^T$

h[m,n]

=



f[m,n]

Discrete derivatives

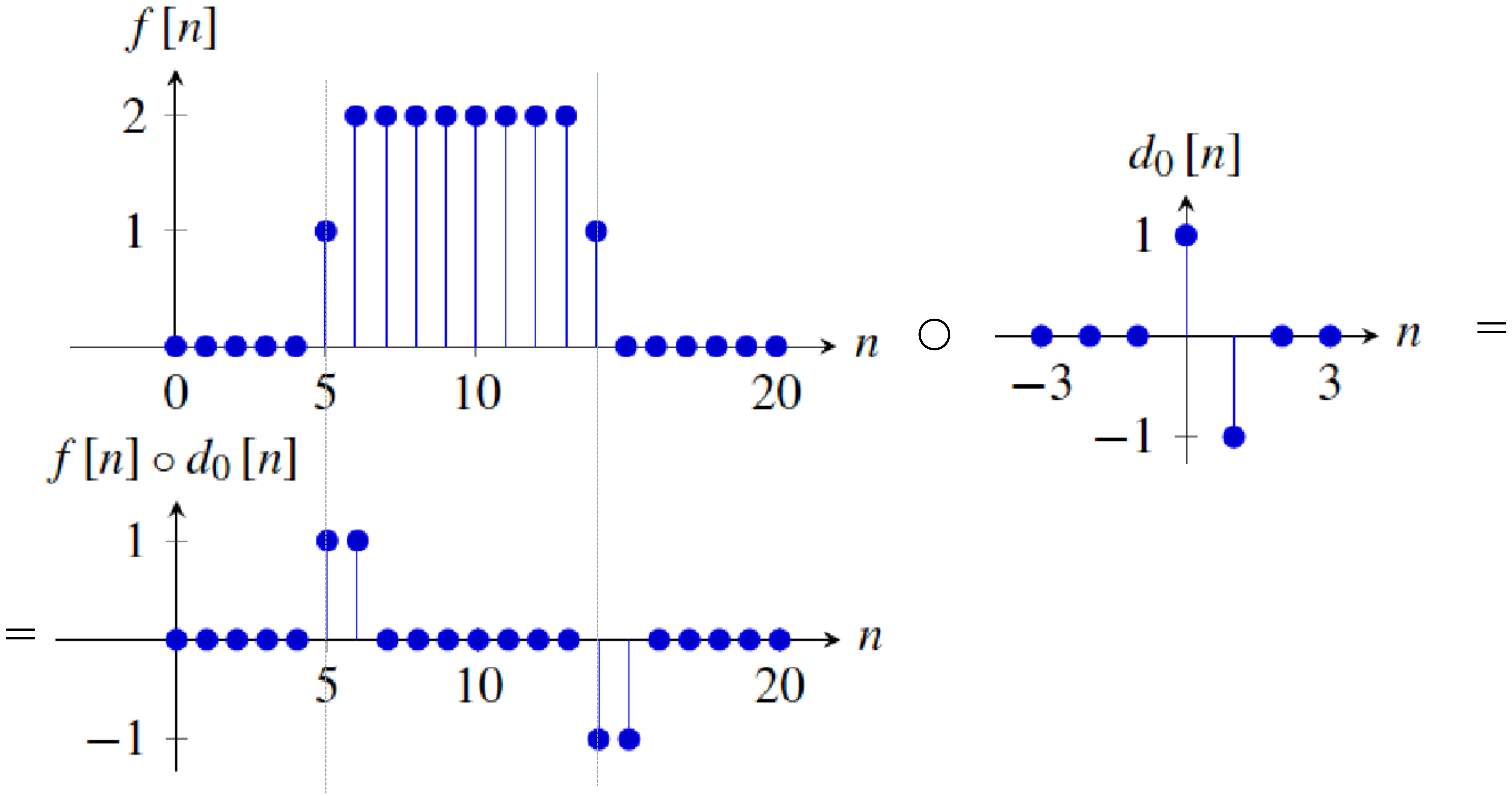
$$d_0 = [1, -1]$$

$$f \circ d_0 = f[n] - f[n-1]$$

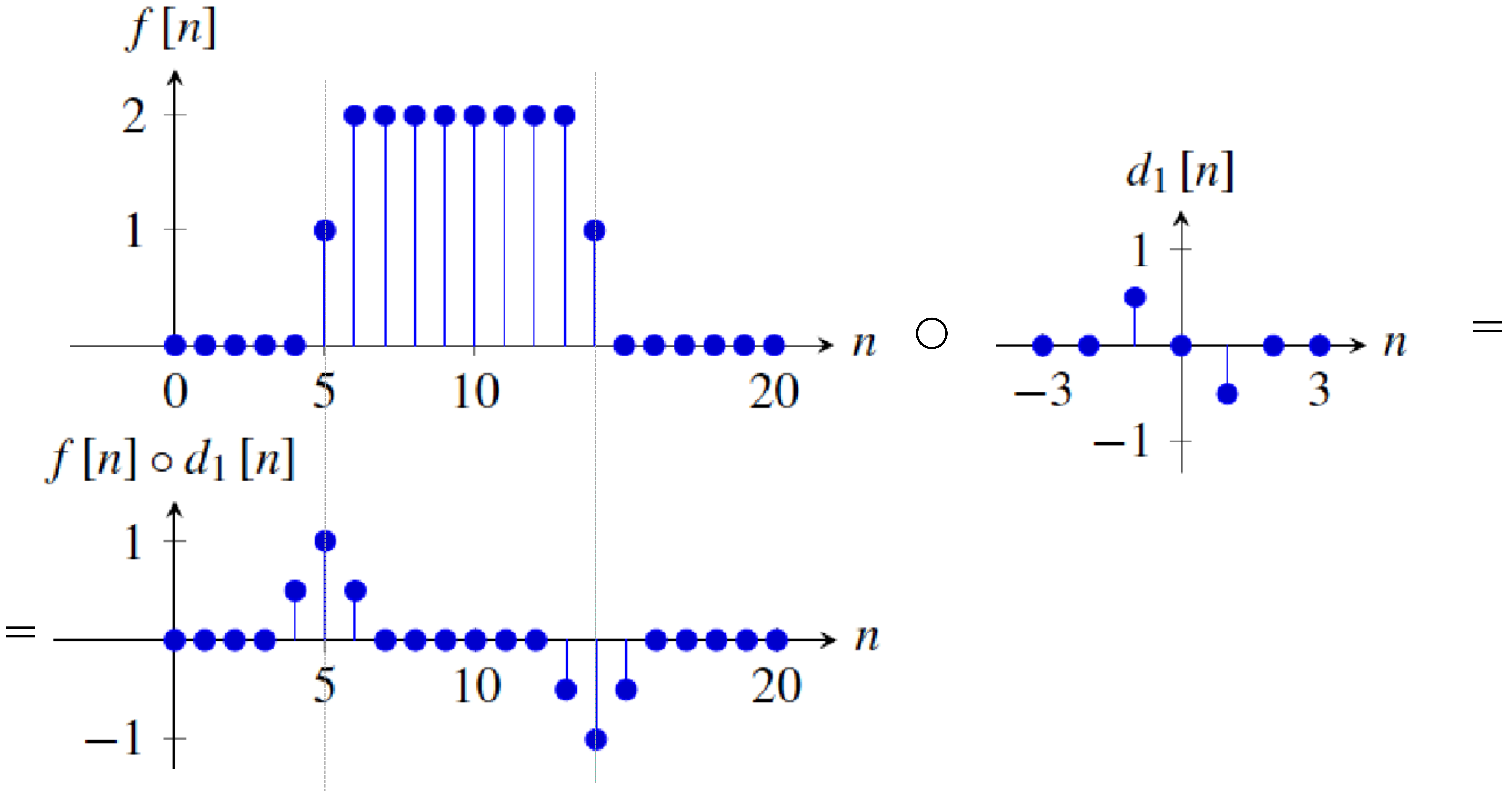
$$d_1 = [1, 0, -1]/2$$

$$f \circ d_1 = \frac{f[n+1] - f[n-1]}{2}$$

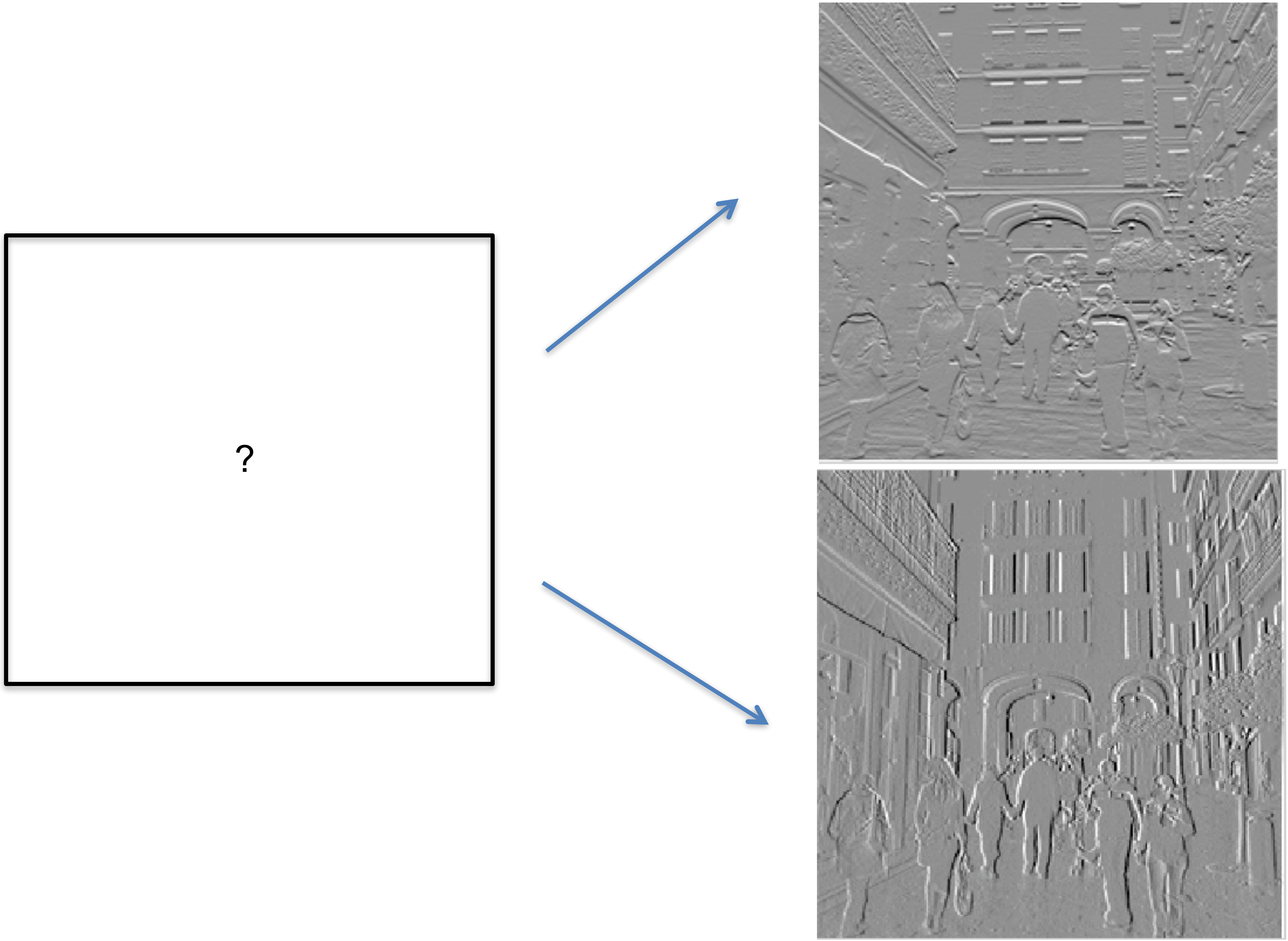
Discrete derivatives



Discrete derivatives

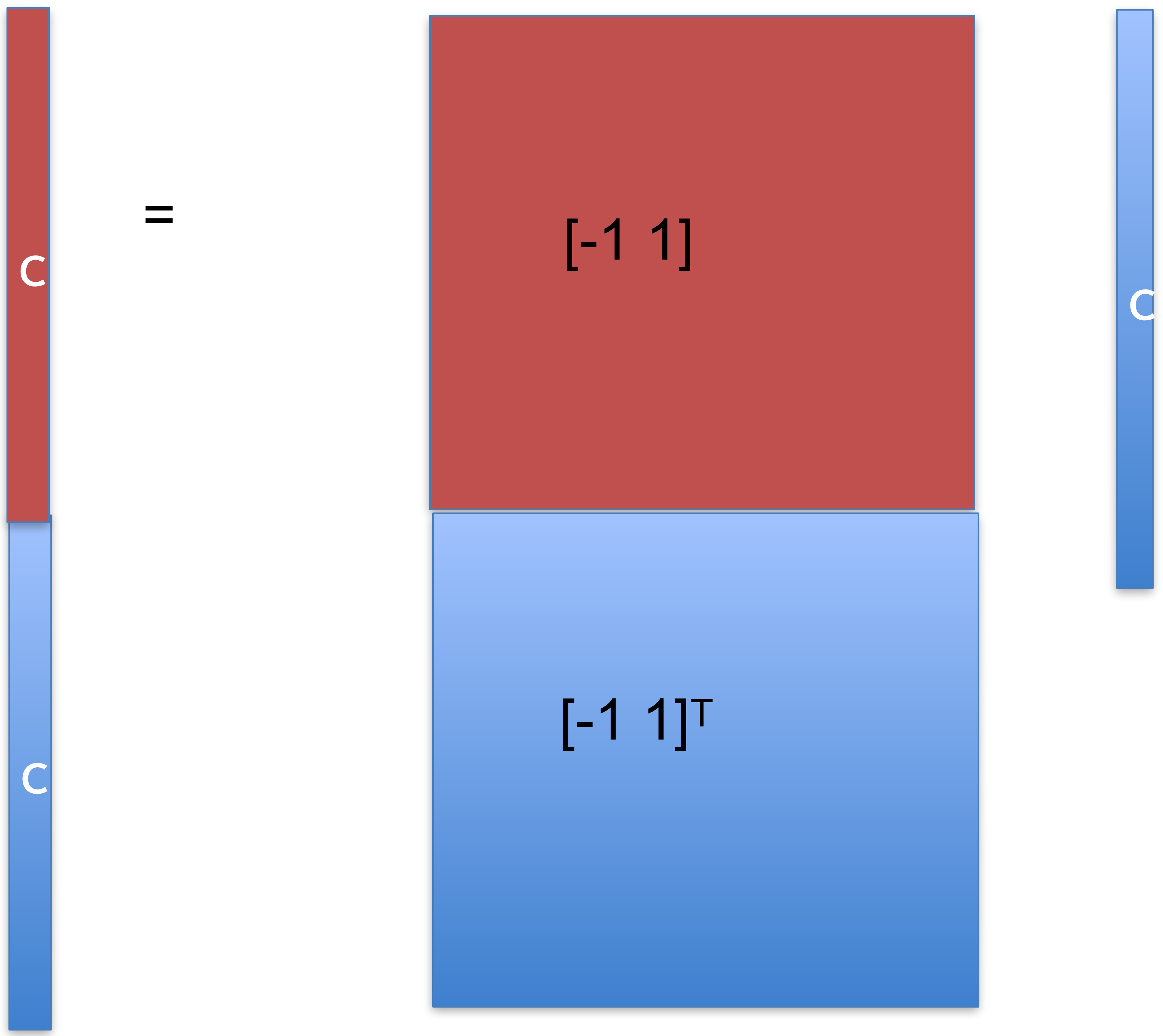


Can you go from the derivatives back to the original image?



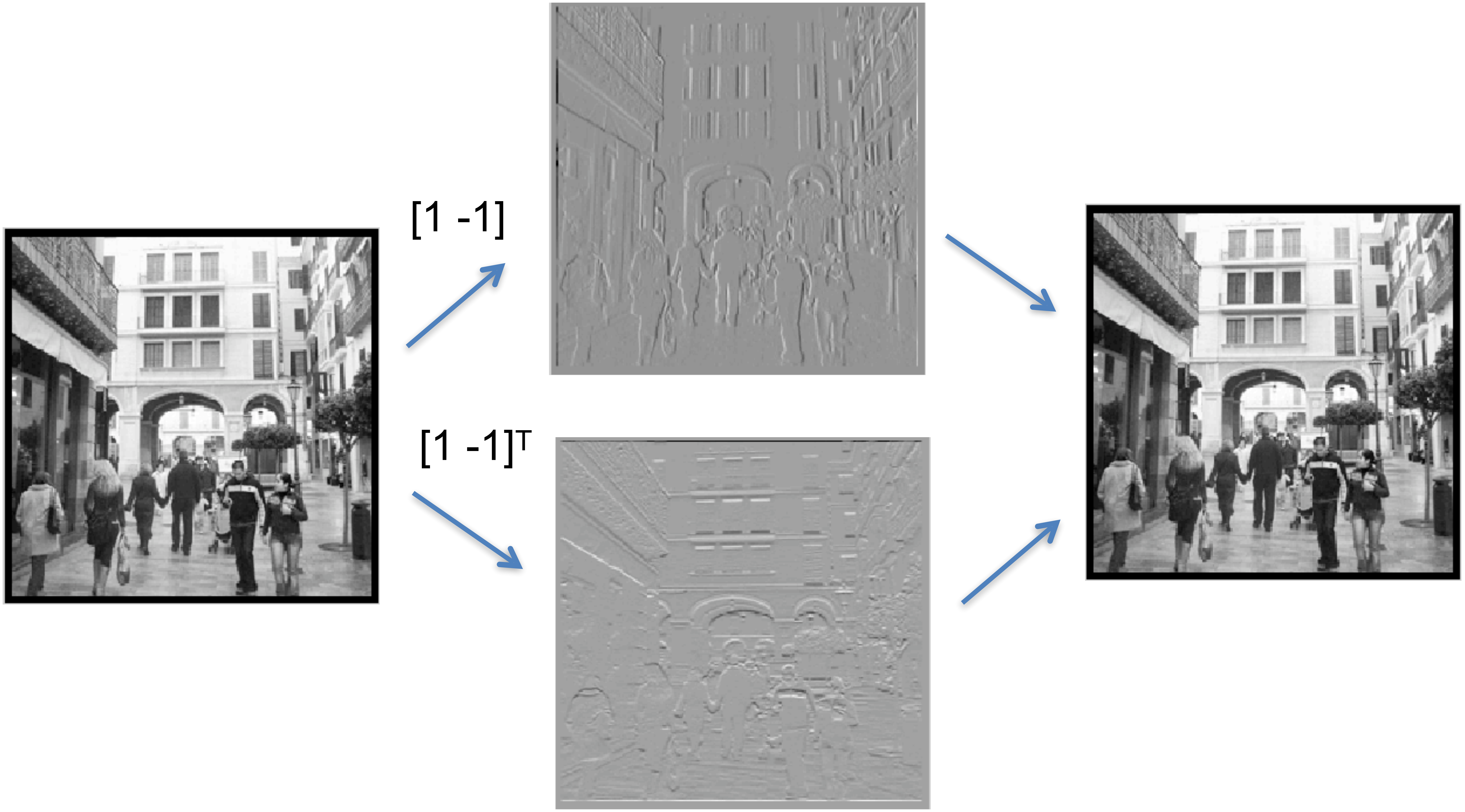
Reconstruction from 2D derivatives

In 2D, we have multiple derivatives (along *n* and *m*)

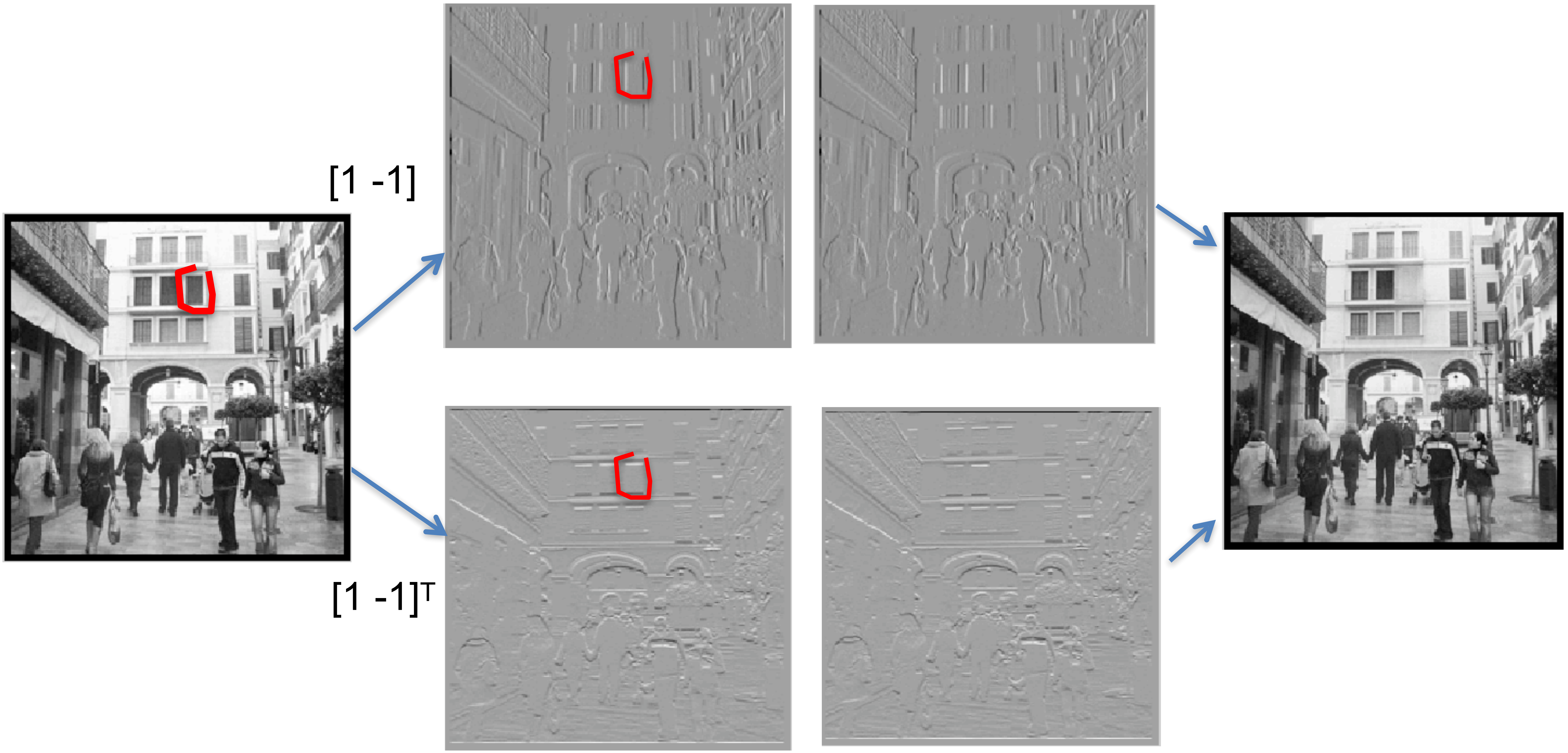


and we compute the pseudo-inverse of the full matrix.

Reconstruction from 2D derivatives



Editing the edge image



Thresholding edges

Issues with image derivatives

- Derivatives are sensitive to noise

- If we consider continuous image derivatives, they might not be defined in some regions (e.g., object boundaries, …)

Derivatives

We want to compute the image derivative:

$$\frac{\partial f(x,y)}{\partial x}$$

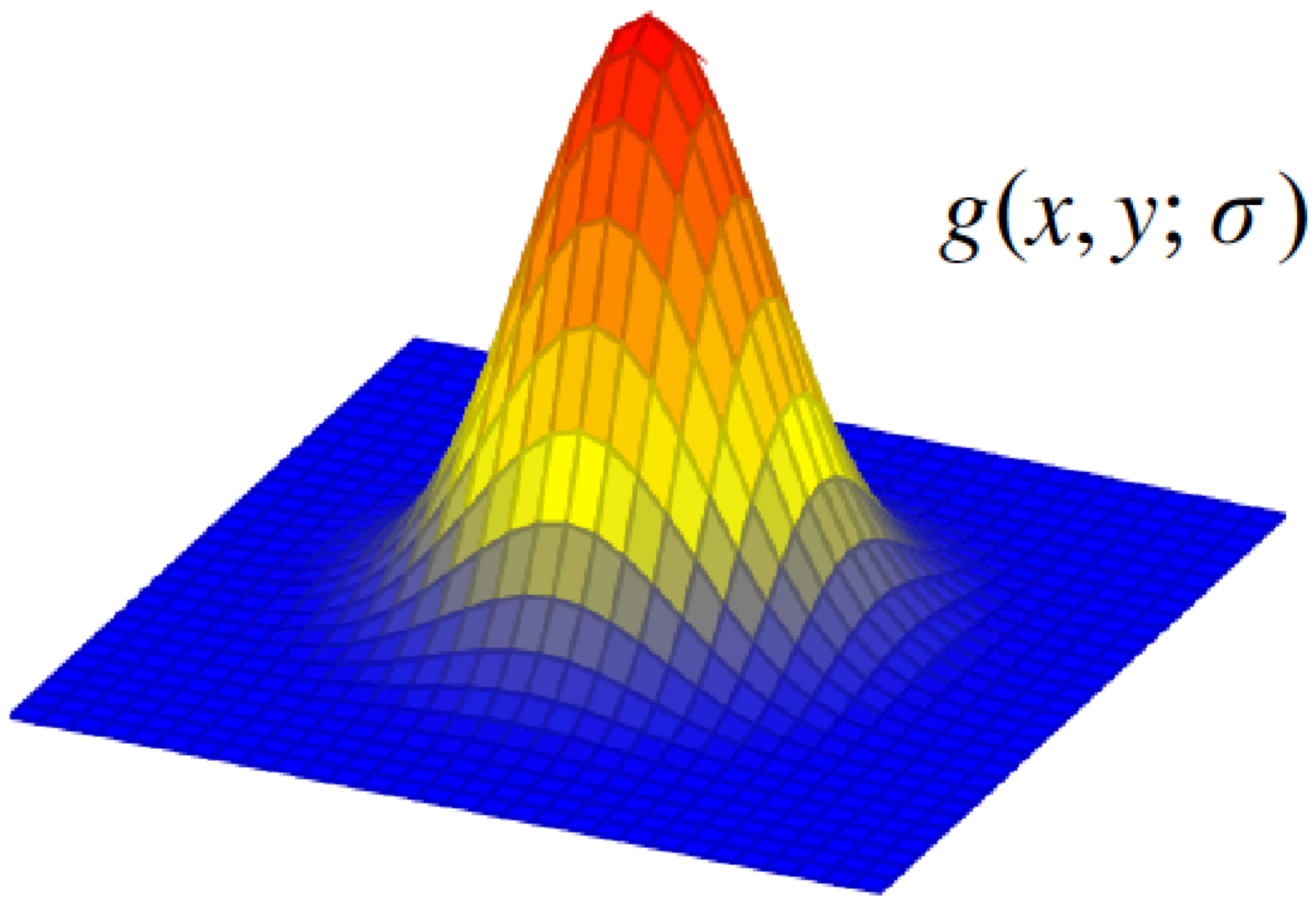
If there is noise, we might want to "smooth" it with a blurring filter

$$\frac{\partial f(x,y)}{\partial x} \circ g(x,y)$$

But derivatives and convolutions are linear and we can move them around:

$$\frac{\partial f(x,y)}{\partial x} \circ g(x,y) = f(x,y) \circ \frac{\partial g(x,y)}{\partial x}$$

Gaussian derivatives



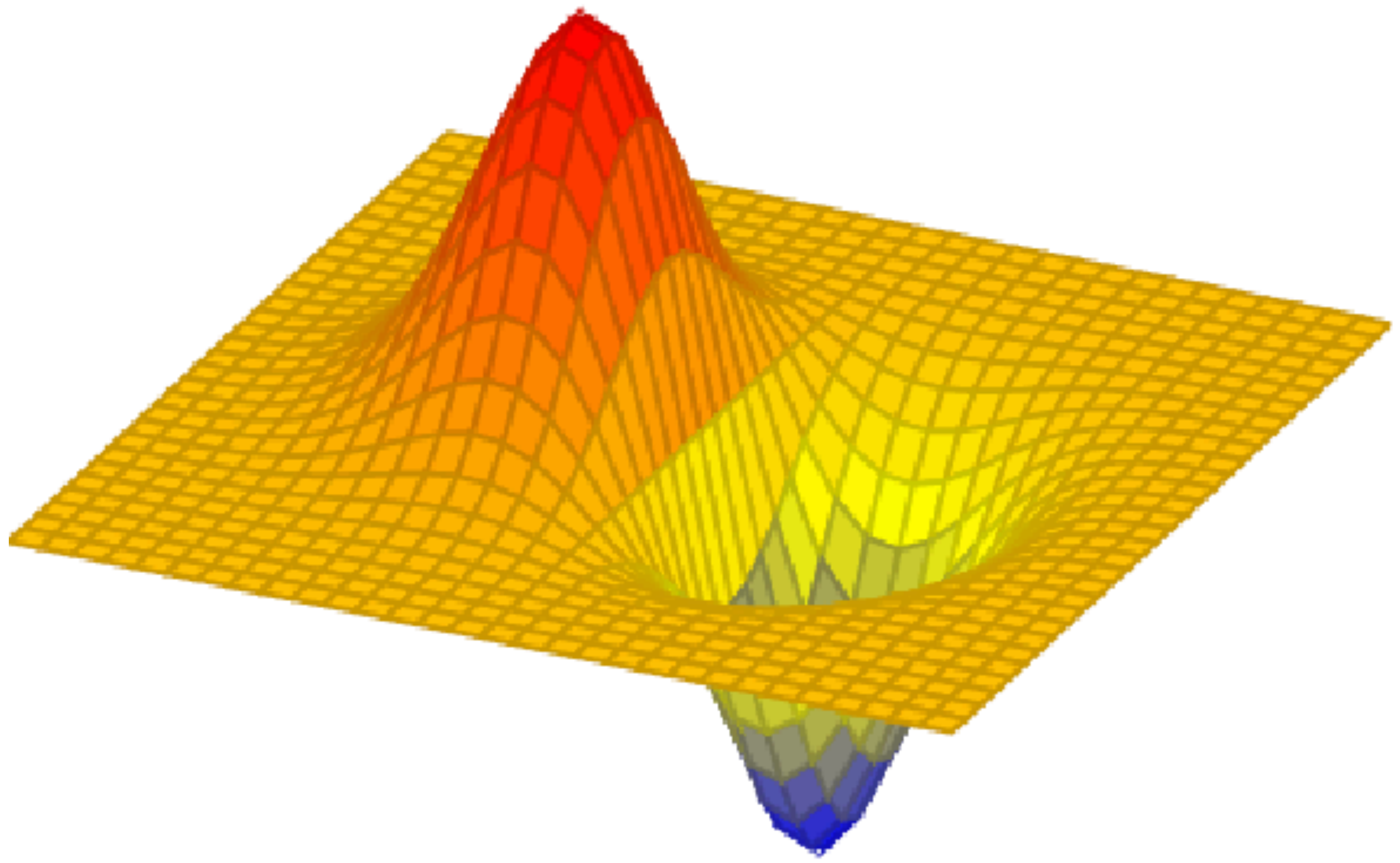
$$g(x, y; \sigma) = \frac{1}{2\pi\sigma^2} \exp - \frac{x^2 + y^2}{2\sigma^2}$$

The continuous derivative is:

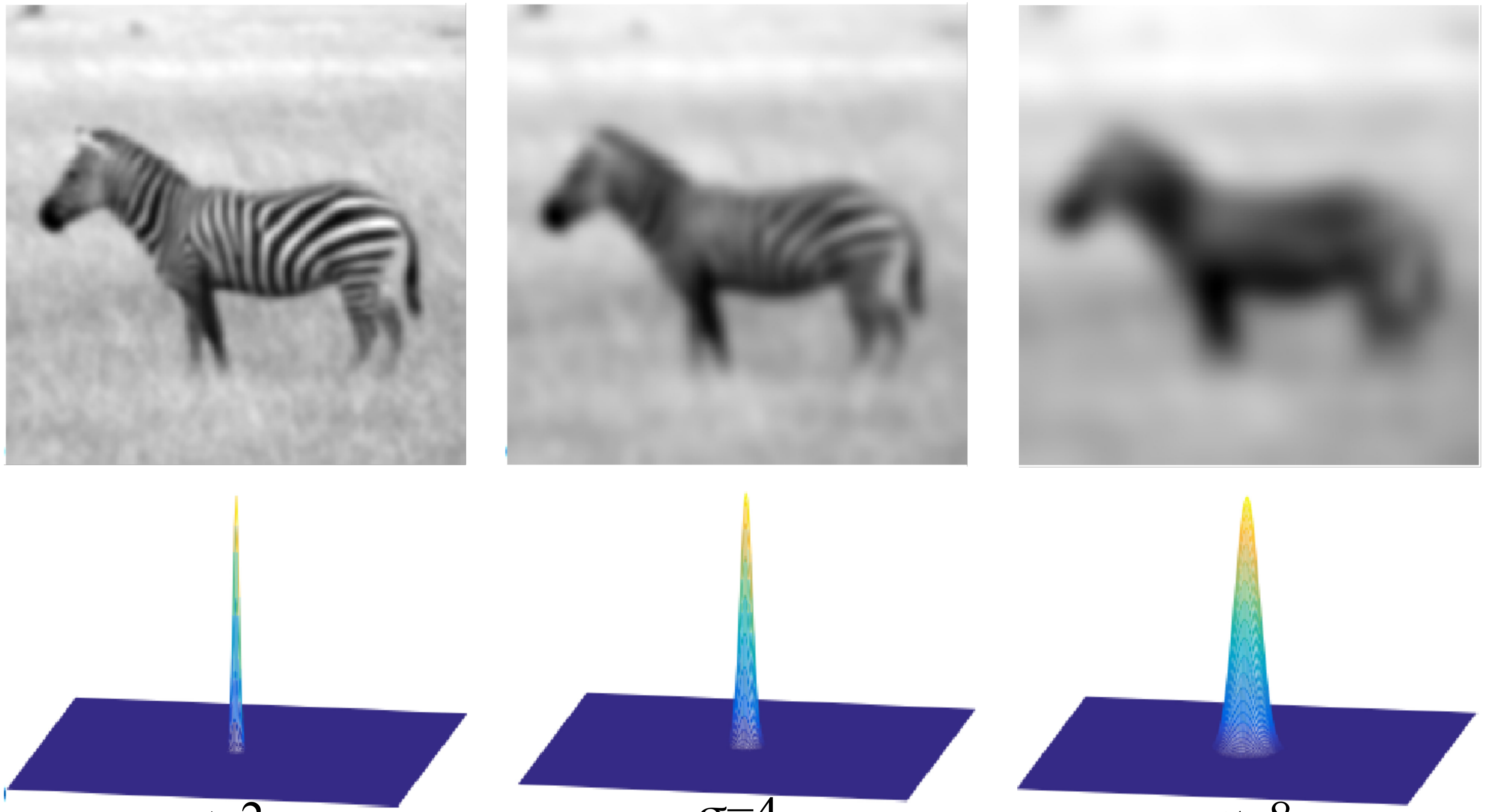
$$g_x(x, y; \sigma) = \frac{\partial g(x, y; \sigma)}{\partial x} =$$

$$= \frac{-x}{2\pi\sigma^4} \exp - \frac{x^2 + y^2}{2\sigma^2}$$

$$= \frac{-x}{\sigma^2} g(x, y; \sigma)$$

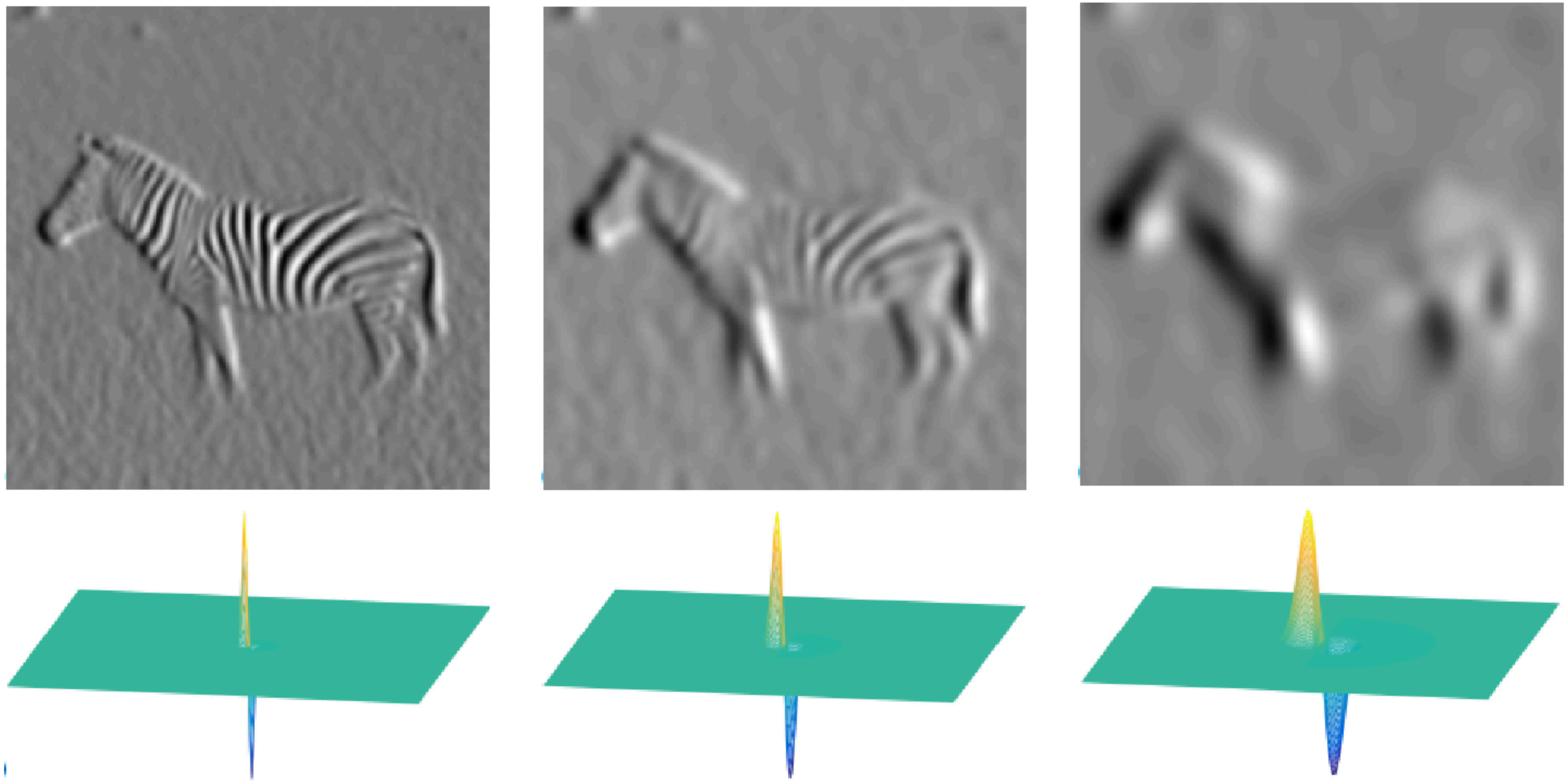


Gaussian Scale



$\sigma=2$ $\sigma=4$ $\sigma=8$

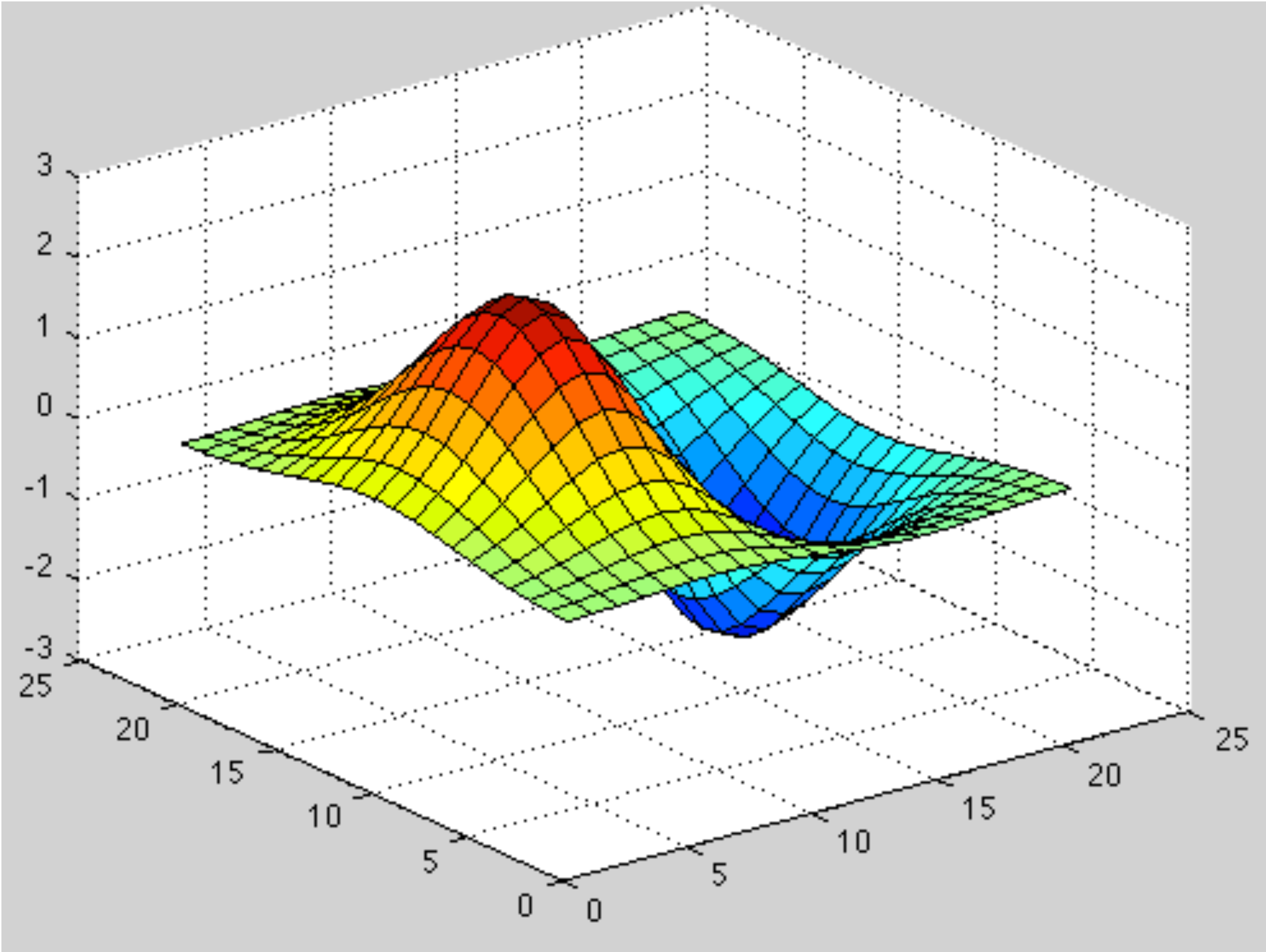
Derivatives of Gaussians: Scale



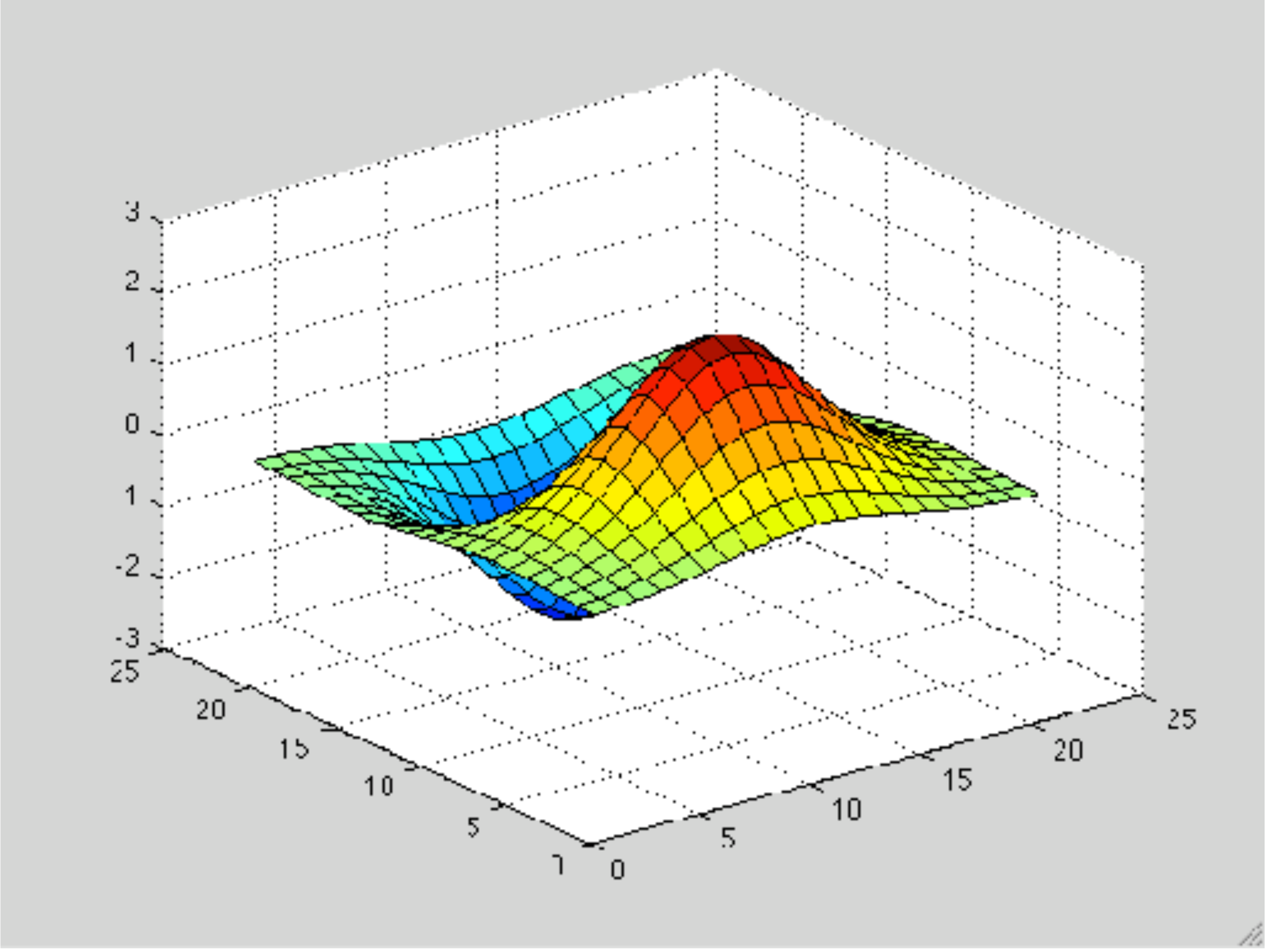
$\sigma=2$ $\sigma=4$ $\sigma=8$

Orientation

$$g_x(x,y) = \frac{\partial g(x,y)}{\partial x} = \frac{-x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

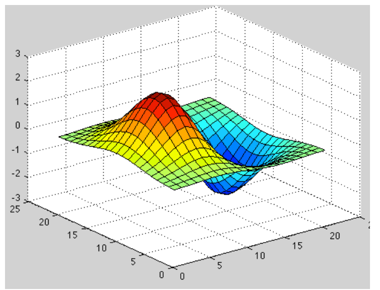


$$g_y(x,y) = \frac{\partial g(x,y)}{\partial y} = \frac{-y}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

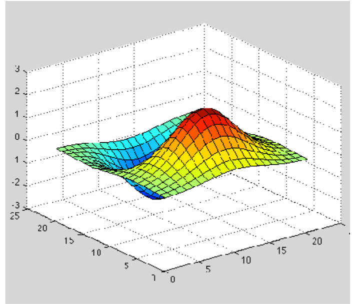


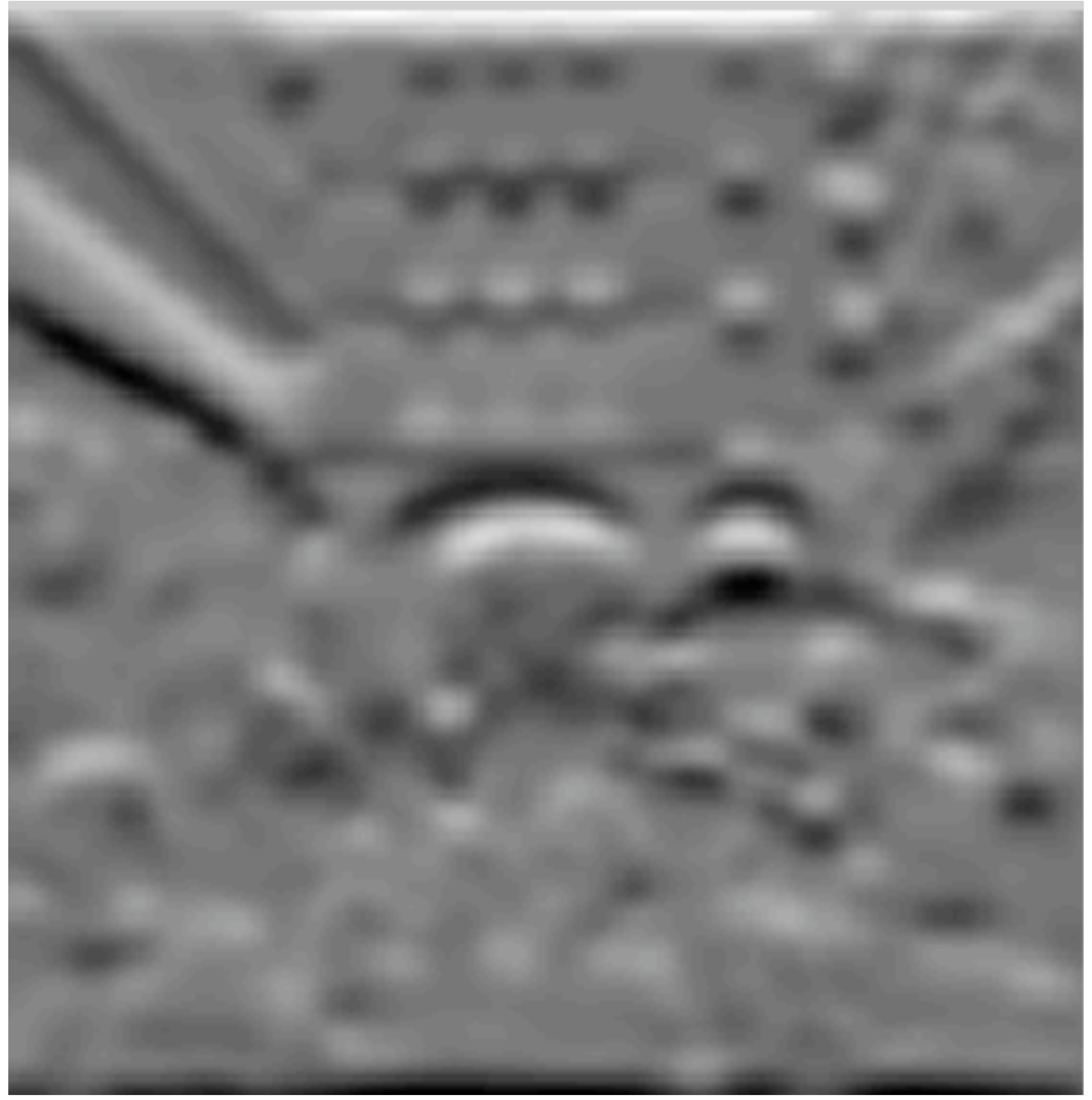
Orientation

$$g_x(x,y) = \frac{\partial g(x,y)}{\partial x} = \frac{-x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



$$g_y(x,y) = \frac{\partial g(x,y)}{\partial y} = \frac{-y}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

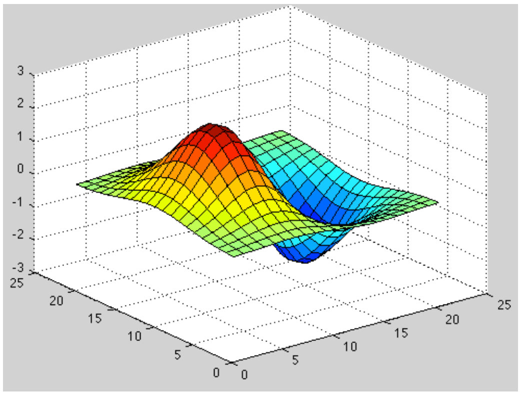




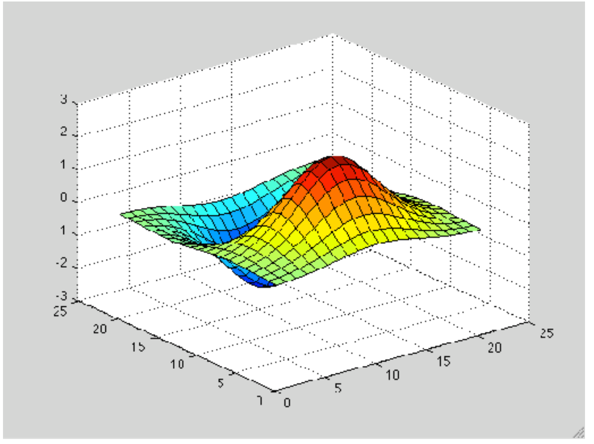
What about other orientations not axis aligned?

Orientation

$$g_x(x,y) = \frac{\partial g(x,y)}{\partial x} = \frac{-x}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



$$g_y(x,y) = \frac{\partial g(x,y)}{\partial y} = \frac{-y}{2\pi\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



The smoothed directional gradient is a linear combination of two kernels

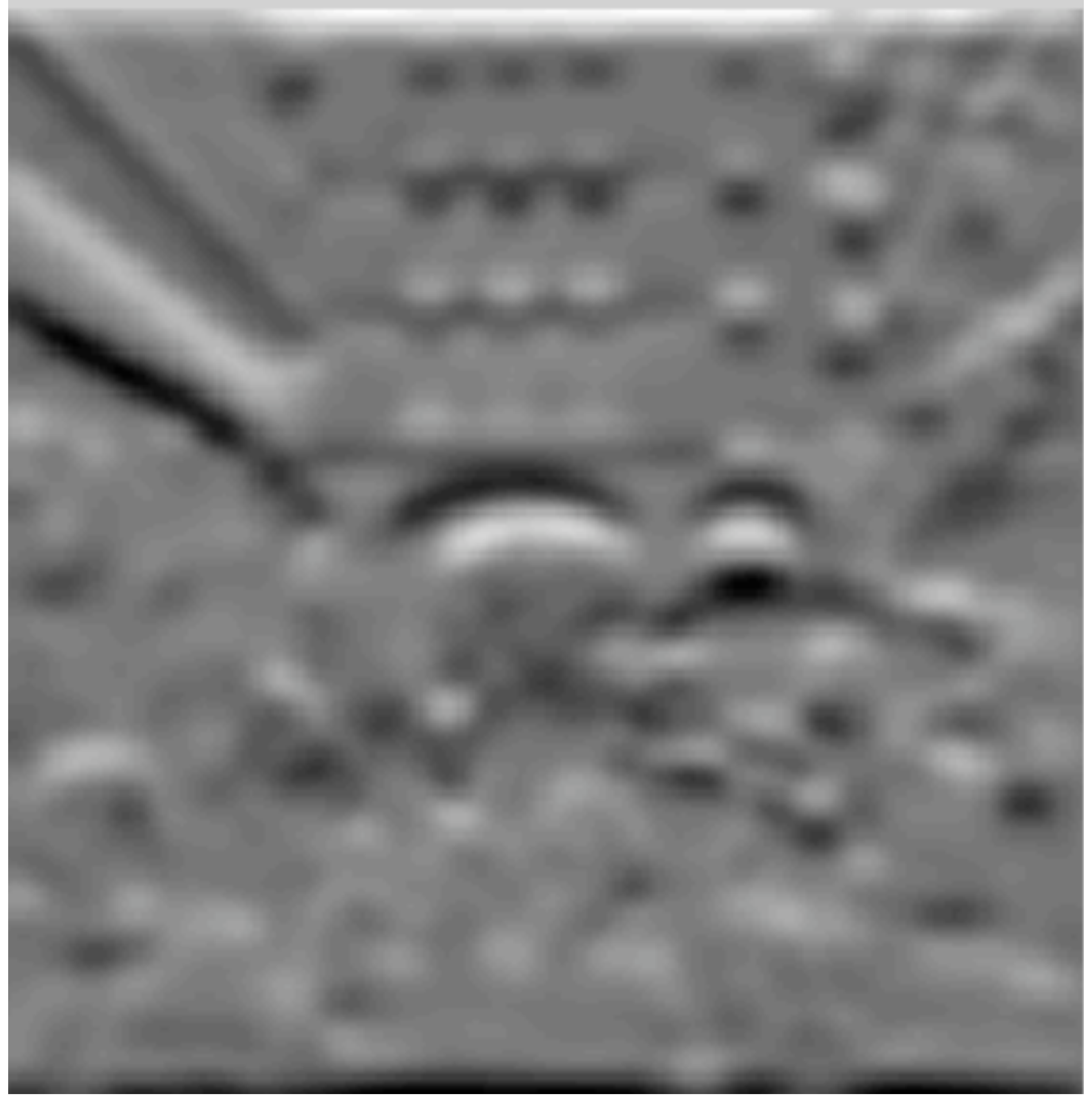
$$u^T \nabla g \otimes I = \left(\cos(\alpha) g_x(x,y) + \sin(\alpha) g_y(x,y)\right) \otimes I(x,y) =$$

Any orientation can be computed as a linear combination of two filtered images

$$= \cos(\alpha) g_x(x,y) \otimes I(x,y) + \sin(\alpha) g_y(x,y) \otimes I(x,y)$$

Steereability of gaussian derivatives, Freeman & Adelson 92

Orientation

$\cos(\alpha)$  $+\sin(\alpha)$  $=$ 

Steereability of gaussian derivatives, Freeman & Adelson 92

Discretization Gaussian derivatives

There are many discrete approximations. For instance, we can take samples of the continuous functions. In practice it is common to use the discrete approximation given by the binomial filters.

Convolving the binomial coefficients with [1, -1]

$$
\begin{array}{c}
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
\quad \xrightarrow{[1, -1]} \quad
\begin{array}{c}
1 \quad -1 \\
1 \quad 0 \quad -1 \\
1 \quad 1 \quad -1 \quad -1 \\
1 \quad 2 \quad 0 \quad -2 \quad -1 \\
1 \quad 3 \quad 2 \quad -2 \quad -3 \quad -1 \\
1 \quad 4 \quad 5 \quad 0 \quad -5 \quad -4 \quad -1
\end{array}
$$

Discretization 2D Gaussian derivatives

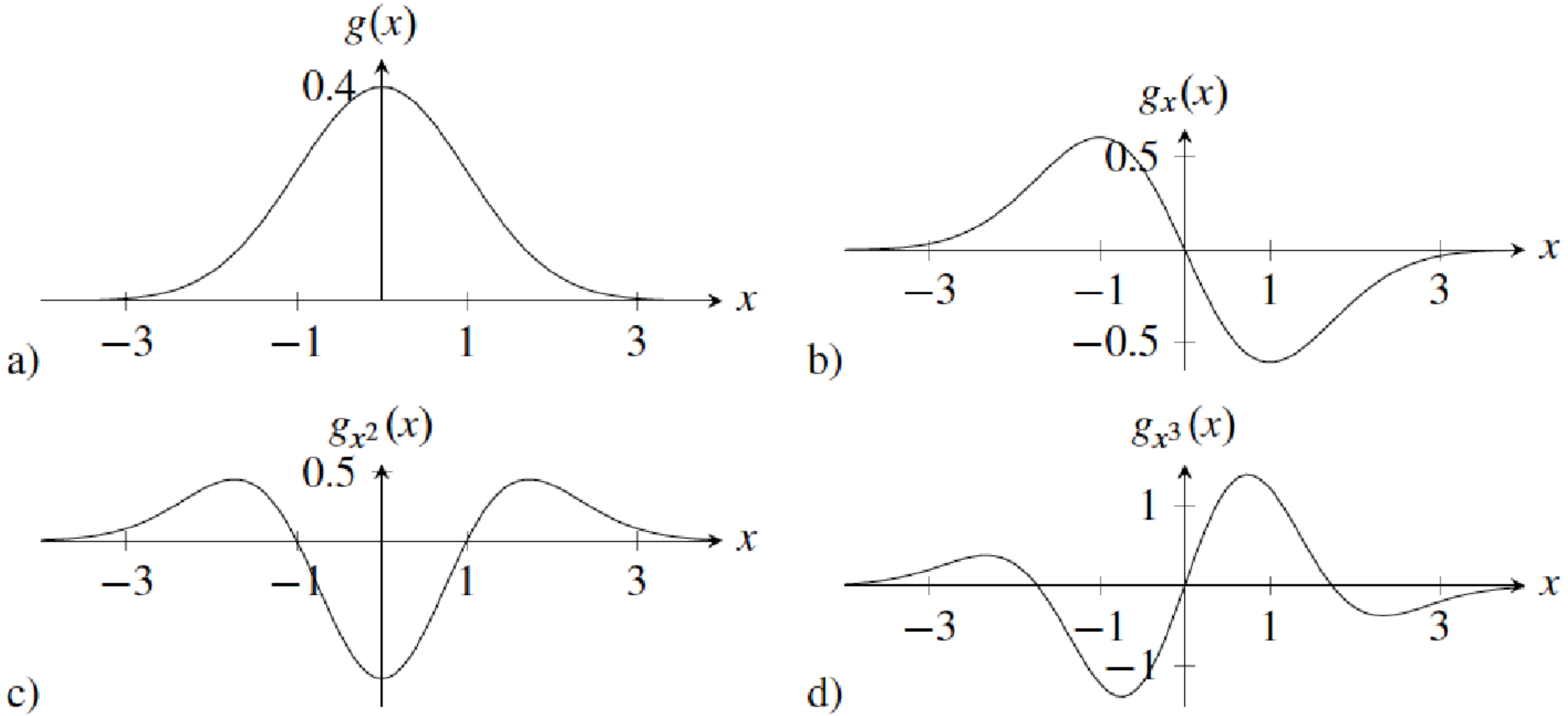
As Gaussians are separable, we can approximate two 1D derivatives and then convolve them.

One example is the Sobel-Feldman operator:

$$Sobel_x = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$Sobel_y = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

n-th order Gaussian derivatives



$$g_{x^n,y^m}(x,y;\sigma) = \frac{\partial^{n+m} g(x,y)}{\partial x^n \partial y^m} = \left(\frac{-1}{\sigma\sqrt{2}}\right)^{n+m} H_n\left(\frac{x}{\sigma\sqrt{2}}\right) H_m\left(\frac{y}{\sigma\sqrt{2}}\right) g(x,y;\sigma)$$

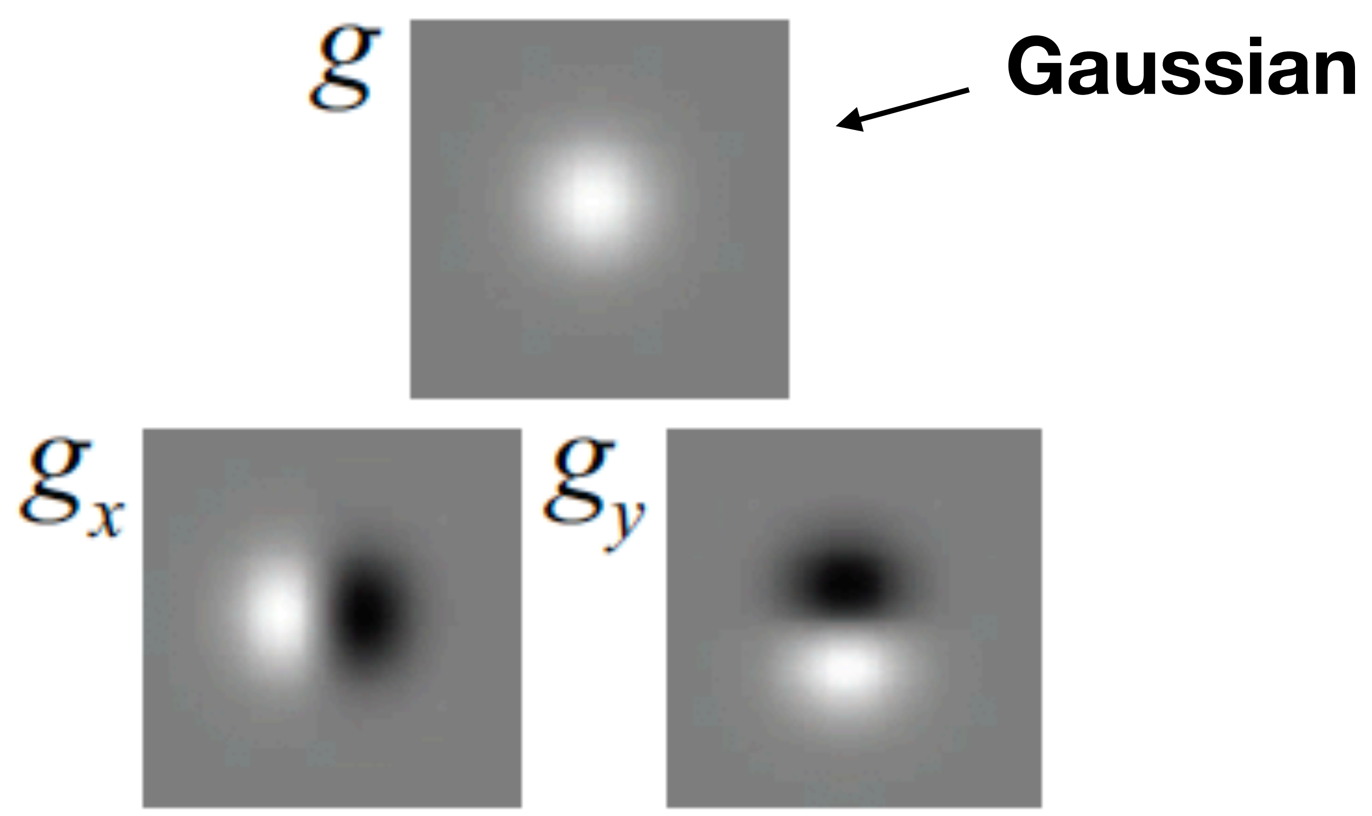
n-th order Gaussian derivatives

g

Gaussian

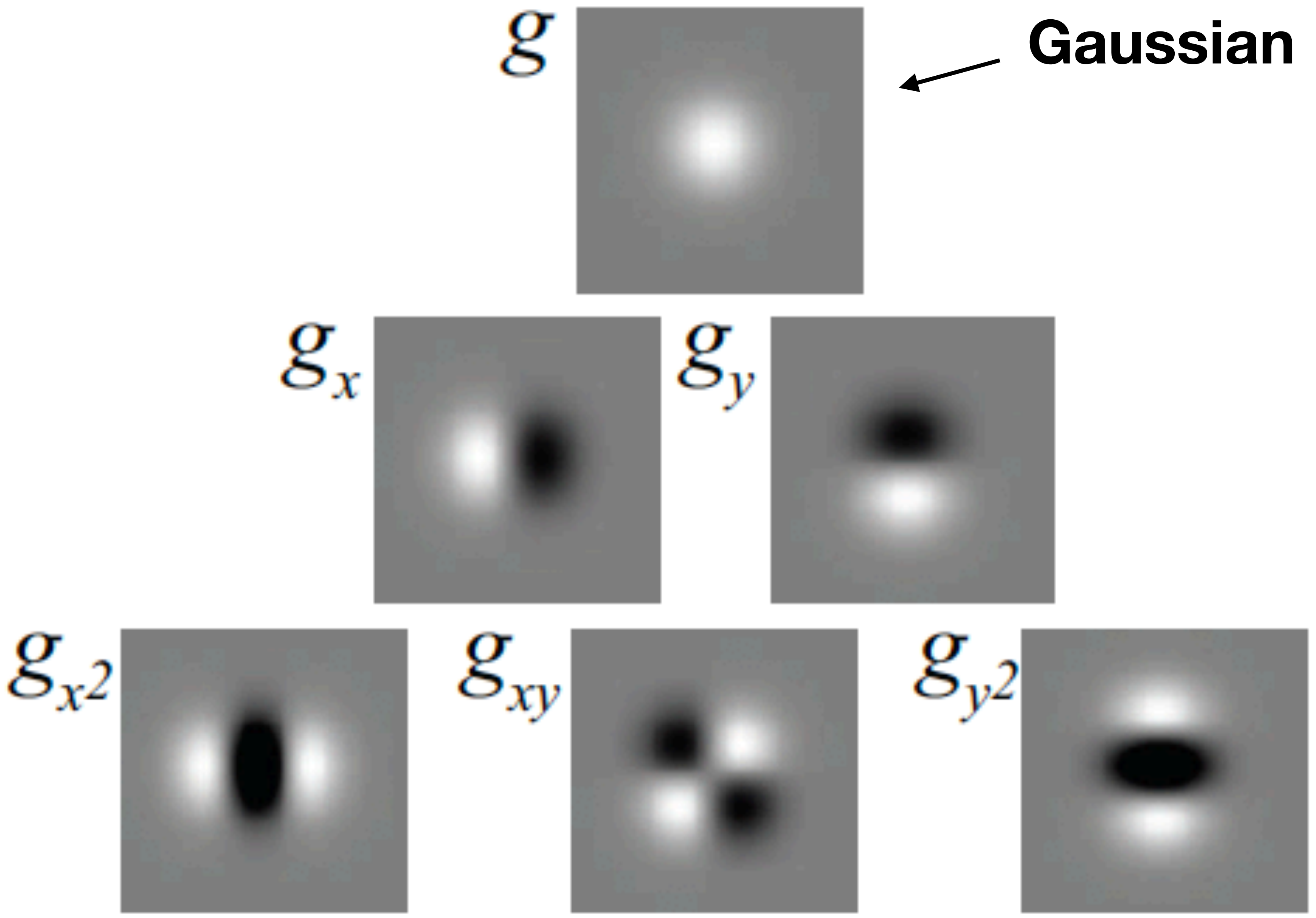
$$g_{x^n,y^m}(x,y;\sigma) = \frac{\partial^{n+m} g(x,y)}{\partial x^n \partial y^m} = \left(\frac{-1}{\sigma\sqrt{2}}\right)^{n+m} H_n\left(\frac{x}{\sigma\sqrt{2}}\right) H_m\left(\frac{y}{\sigma\sqrt{2}}\right) g(x,y;\sigma)$$

n-th order Gaussian derivatives



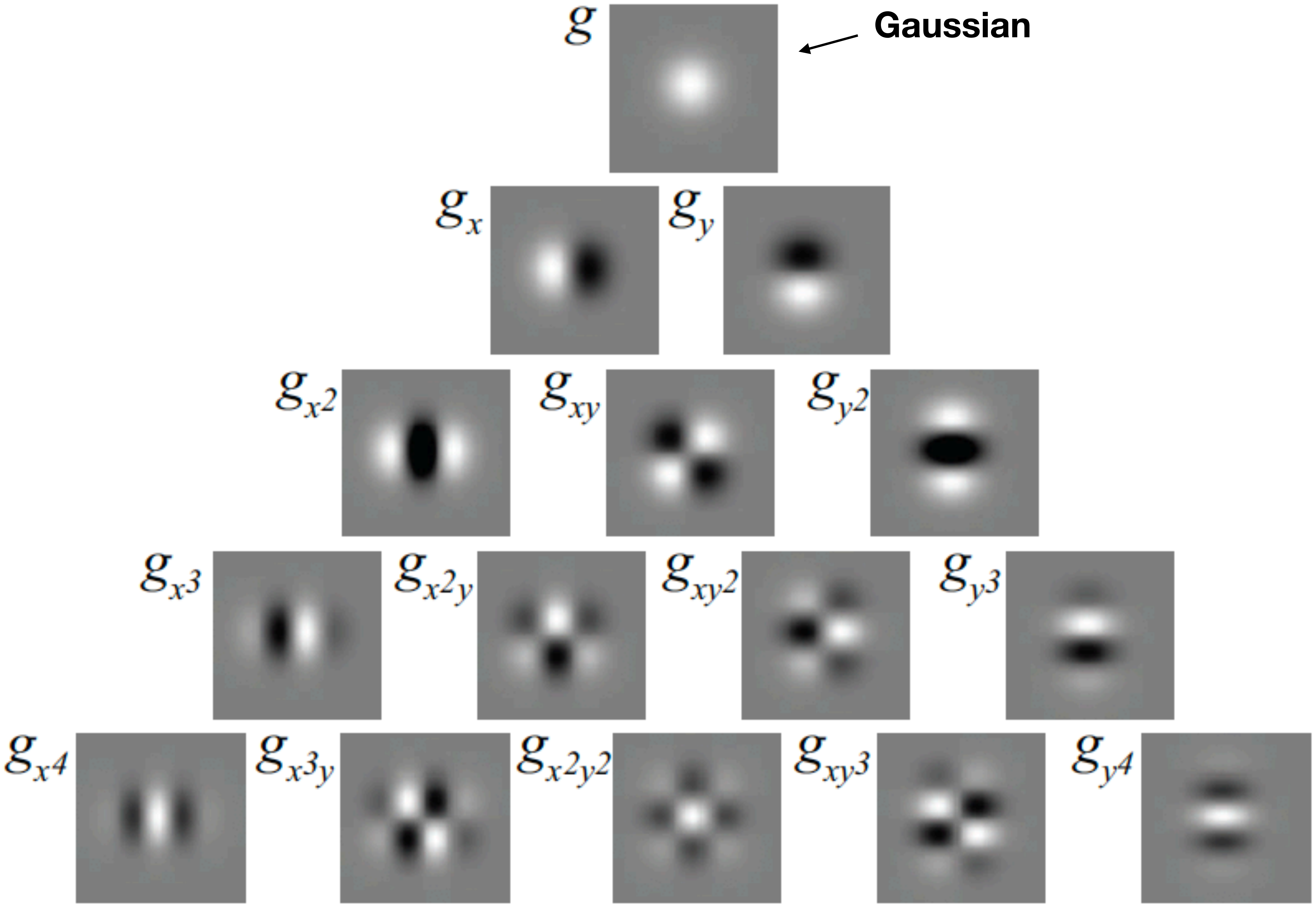
$$g_{x^n,y^m}(x, y; \sigma) = \frac{\partial^{n+m} g(x, y)}{\partial x^n \partial y^m} = \left(\frac{-1}{\sigma\sqrt{2}}\right)^{n+m} H_n\left(\frac{x}{\sigma\sqrt{2}}\right) H_m\left(\frac{y}{\sigma\sqrt{2}}\right) g(x, y; \sigma)$$

n-th order Gaussian derivatives



$$g_{x^n,y^m}(x,y;\sigma) = \frac{\partial^{n+m} g(x,y)}{\partial x^n \partial y^m} = \left(\frac{-1}{\sigma\sqrt{2}}\right)^{n+m} H_n\left(\frac{x}{\sigma\sqrt{2}}\right) H_m\left(\frac{y}{\sigma\sqrt{2}}\right) g(x,y;\sigma)$$

n-th order Gaussian derivatives



$$g_{x^n,y^m}(x,y;\sigma) = \frac{\partial^{n+m} g(x,y)}{\partial x^n \partial y^m} = \left(\frac{-1}{\sigma\sqrt{2}}\right)^{n+m} H_n\left(\frac{x}{\sigma\sqrt{2}}\right) H_m\left(\frac{y}{\sigma\sqrt{2}}\right) g(x,y;\sigma)$$

Image sharpening filter

Image sharpening filter

Subtract away the blurred components of the image:

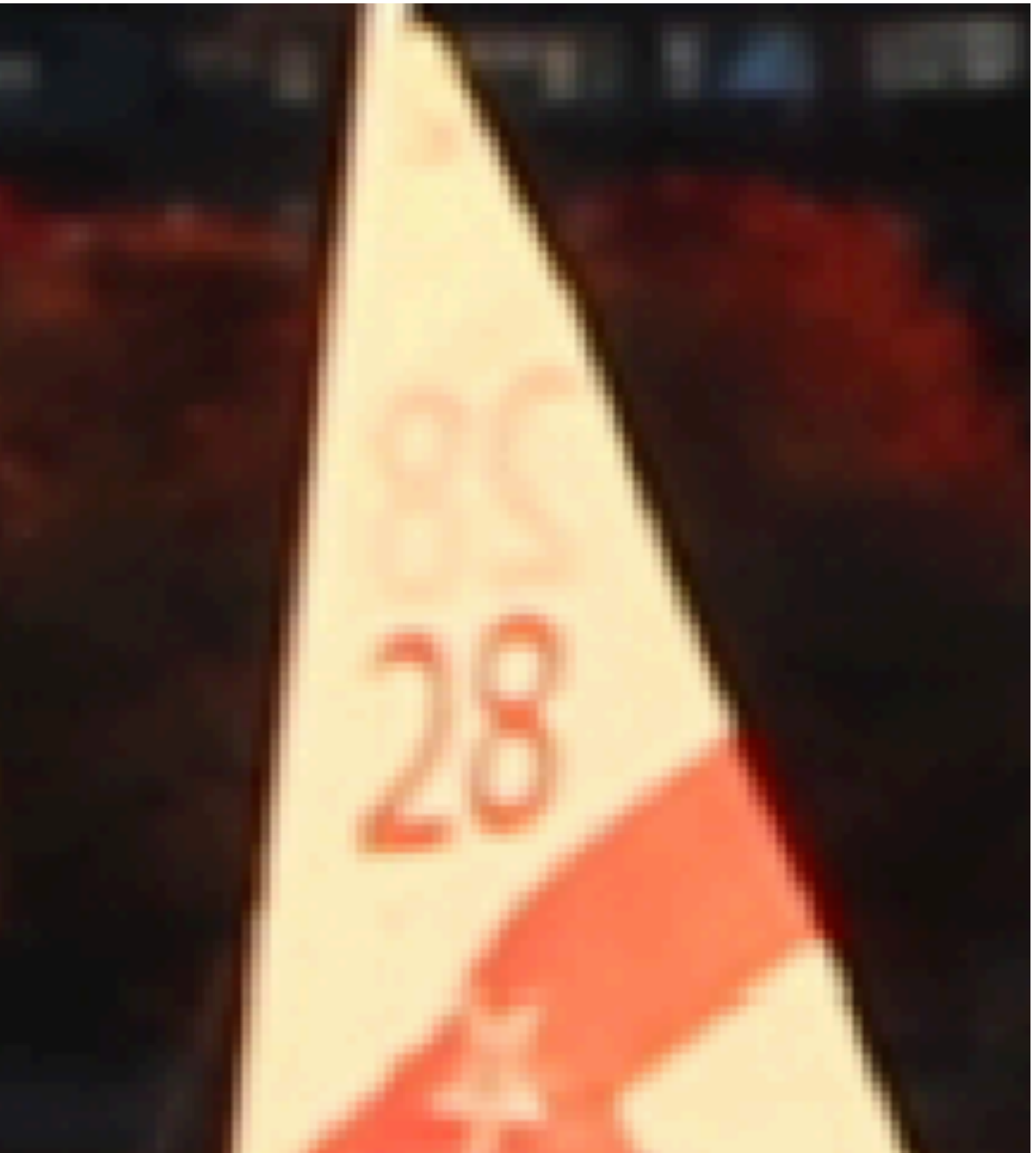
$$\text{sharpening filter} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

This filter has an overall DC component of 1. It de-emphasizes the blur component of the image (low spatial frequencies).

Input image

Sharpened

Input image



Sharpened

Input image