# Lecture 6
# Motion Filtering and Sampling

6.869/6.819 Advances in Computer Vision

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spring 2022

Feb 17, 2022

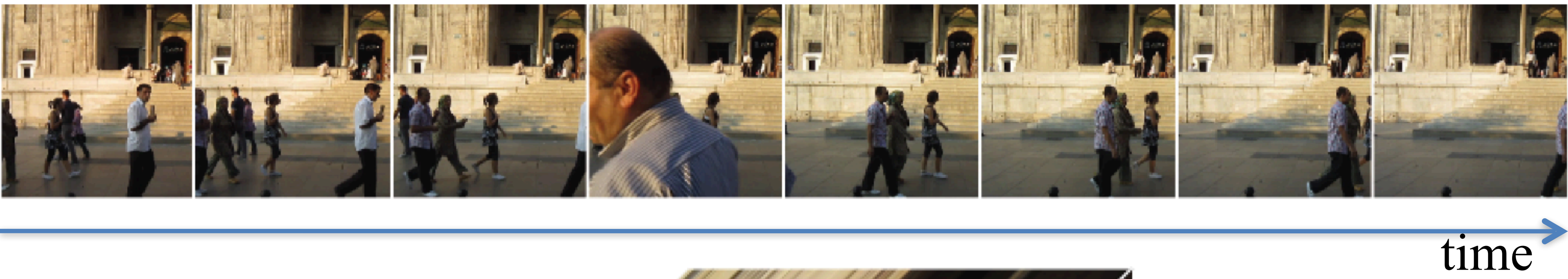
# Today's content

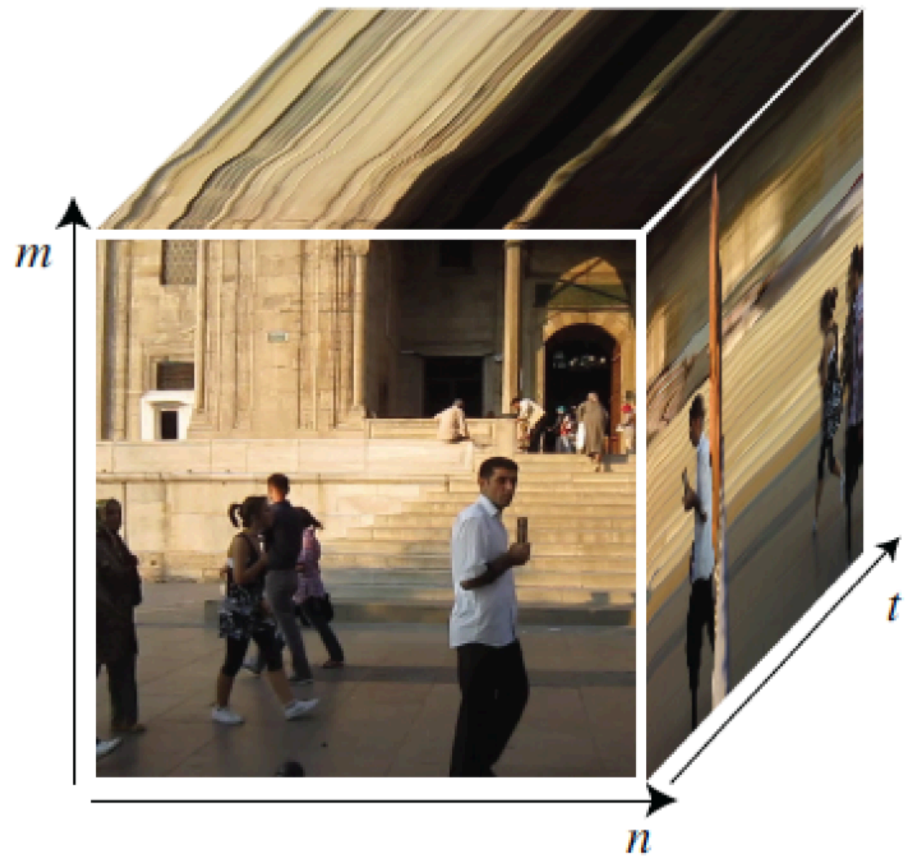
- **Temporal filtering**—what sorts of things can you do with it?
  - picking out objects moving with a certain velocity.
- **Gabor filters** and quadrature pairs.
- Measuring or synthesizing motion.
- **Aliasing**
- **Motion illusion**, involving aliasing, addressing whether humans match spatial patterns, or use temporal filters, to measure motion.
- If there's time: using temporal filtering to remove objects moving with a certain velocity.

# Temporal filtering

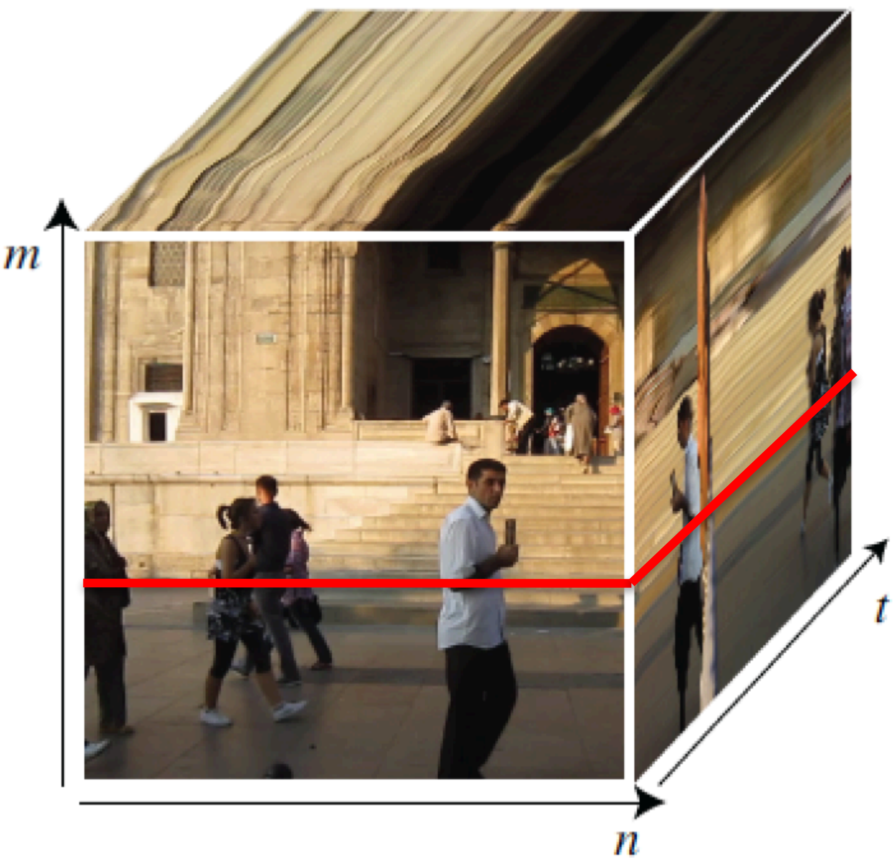
why filter videos over time?

# Sequences

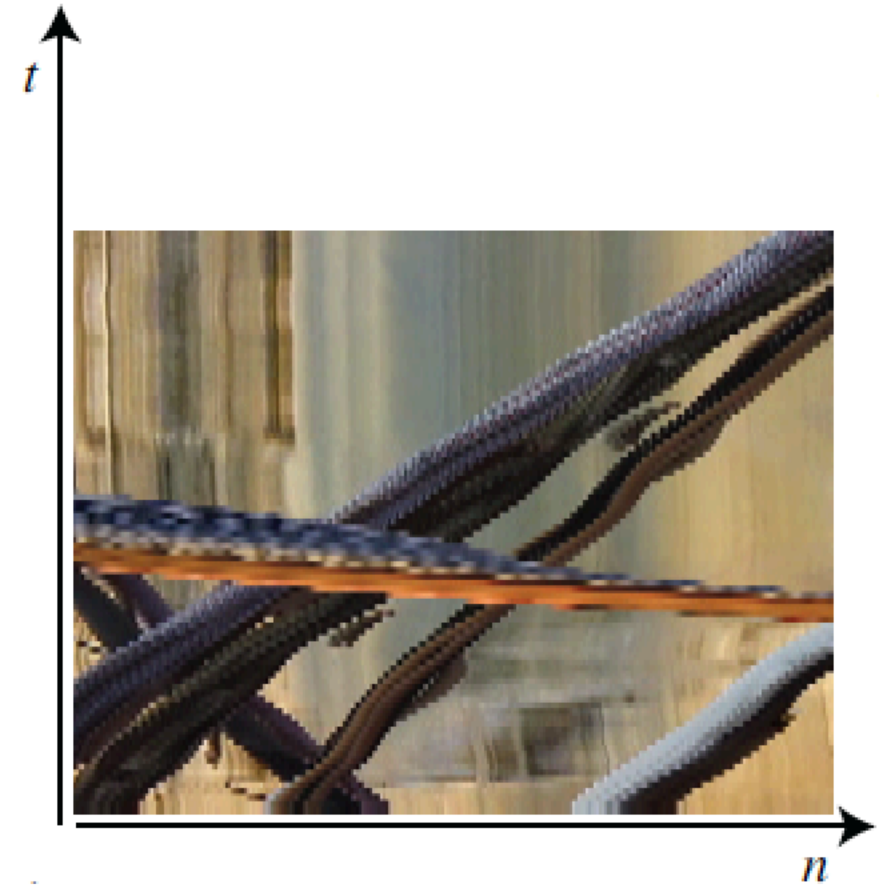


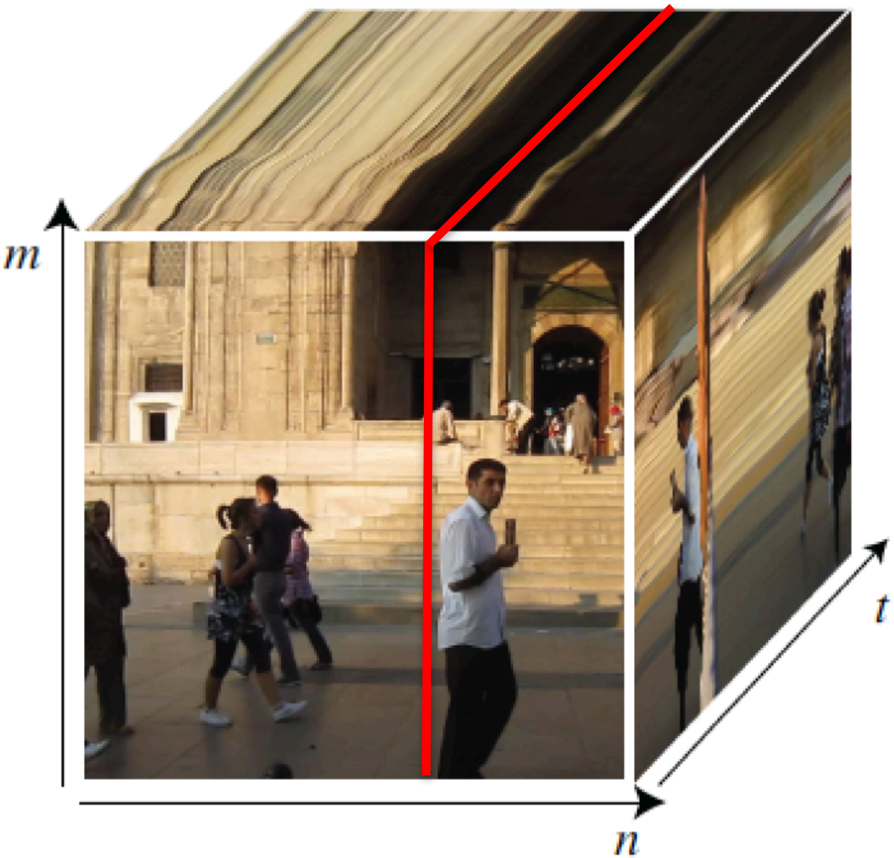
# Sequences




Cube size = 128x128x90



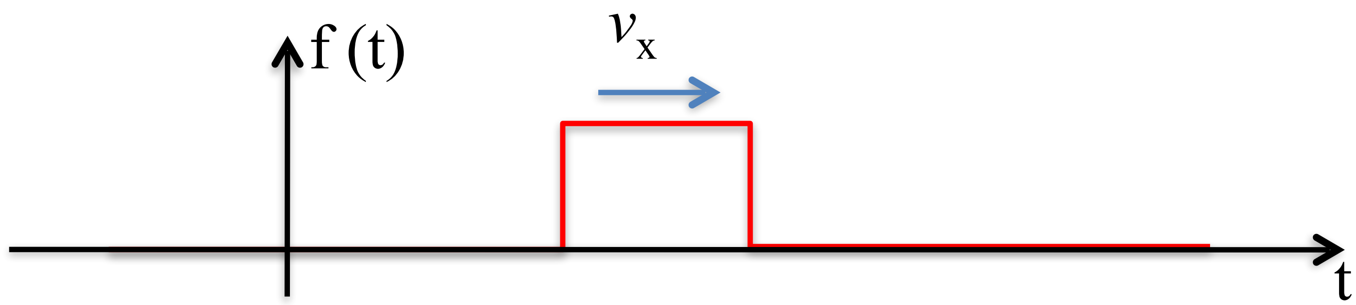
# Sequences

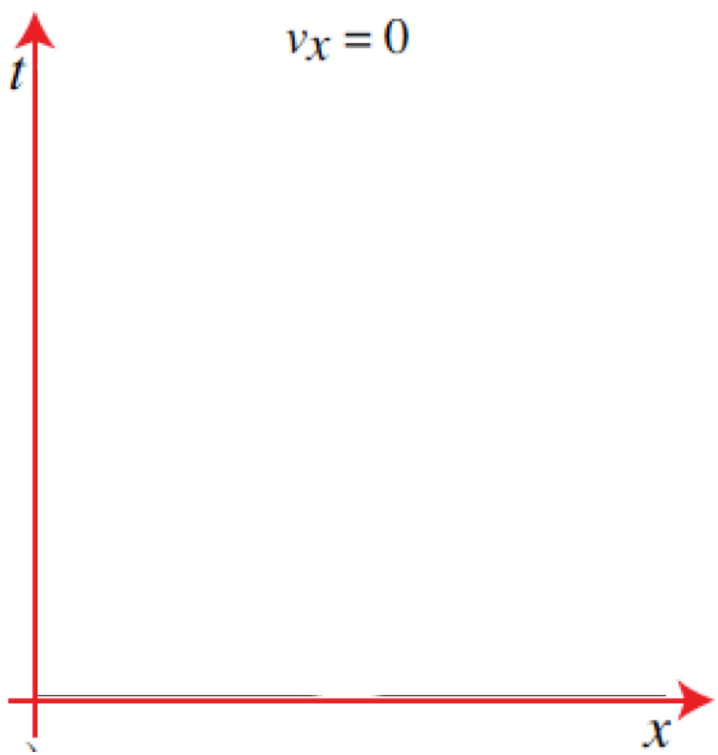


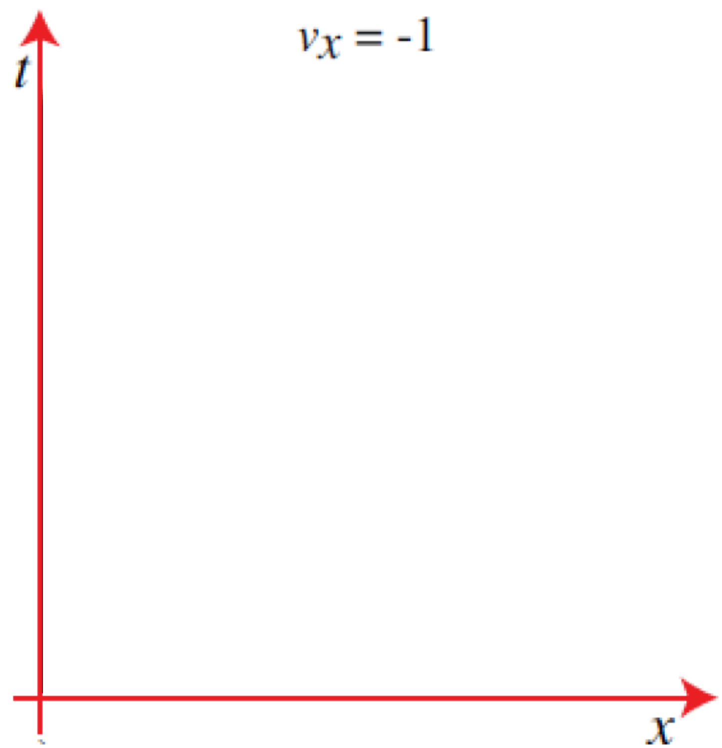


Cube size = 128x128x90

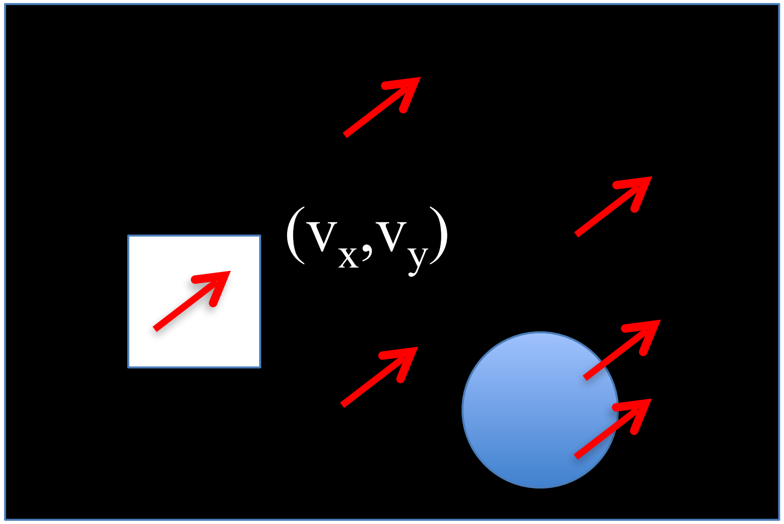
# A box moving with speed $v_x$



# Global constant motion




A global motion of the image can be written as:

$$f(x,y,t) = f_0(x - v_x t, y - v_y t)$$

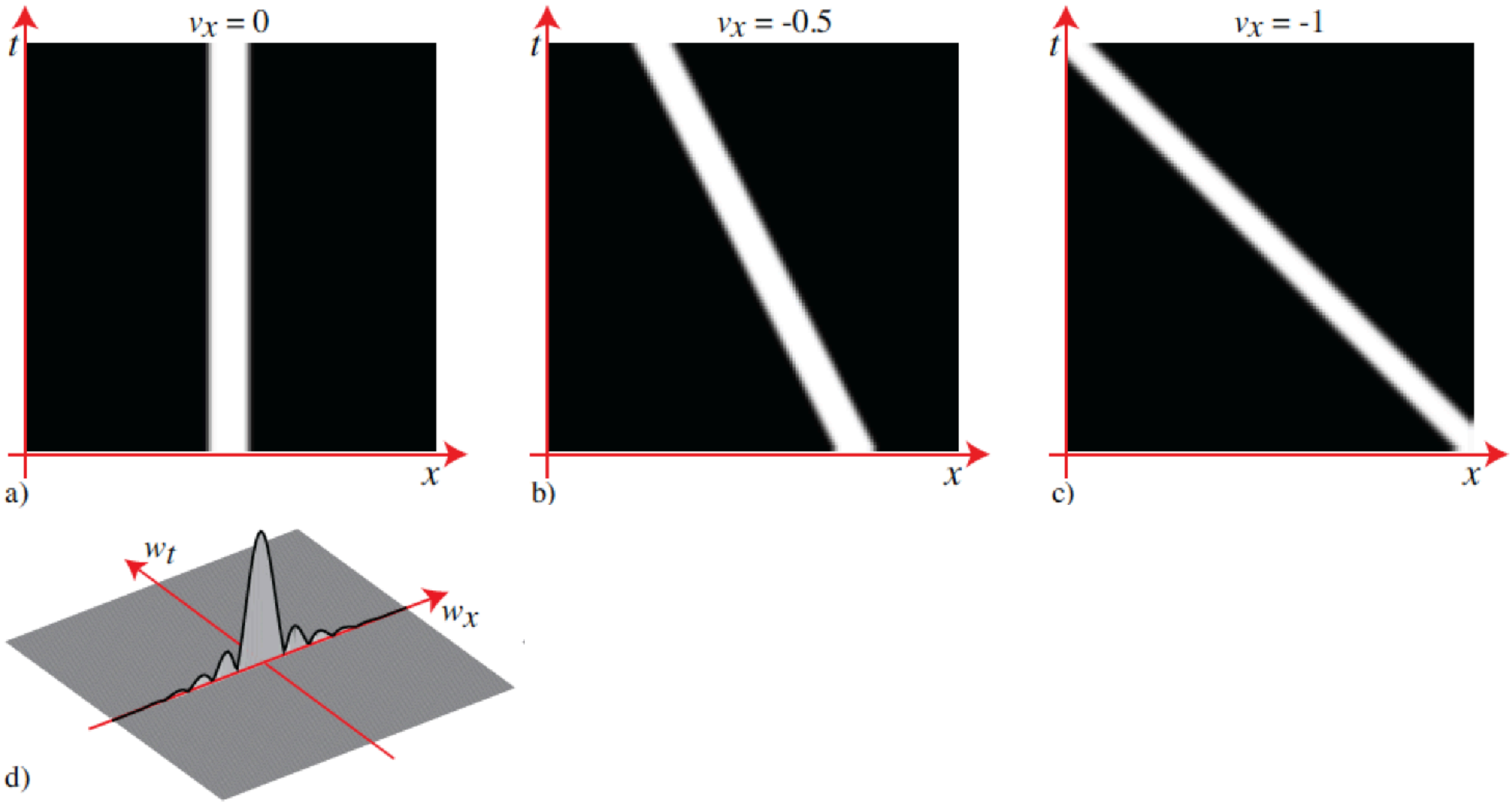
Where:

$$f_0(x,y) = f(x,y,0)$$

$$f(x,y,t) = f_0(x - v_x t, y - v_y t)$$

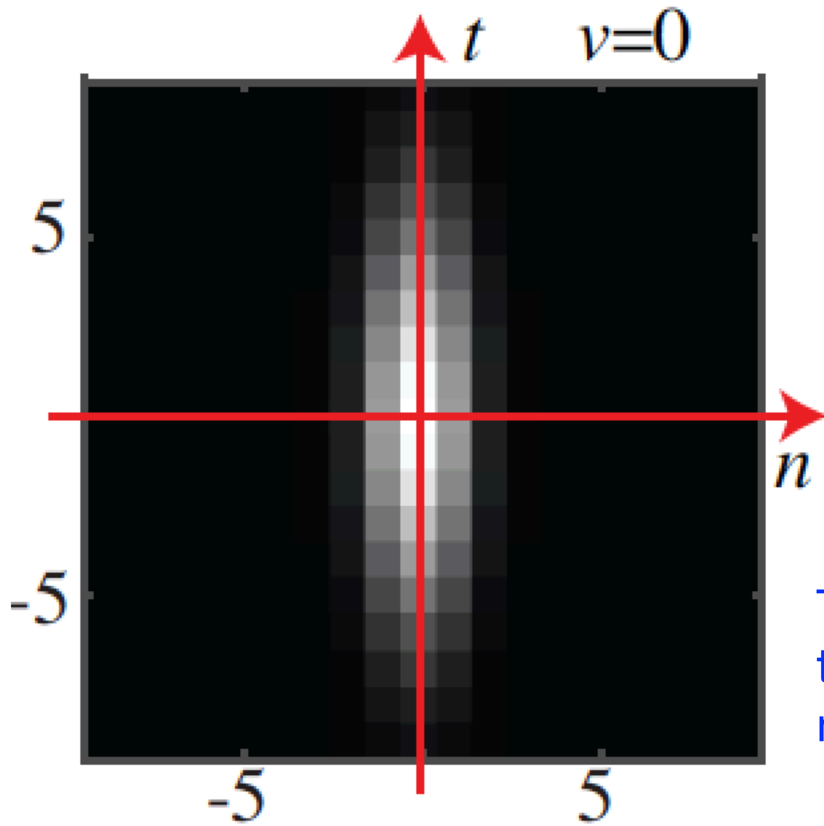
$$\updownarrow$$

$$F(w_x, w_y, w_t) = F_0(w_x, w_y)\delta(w_t + v_x w_x + v_y w_y)$$



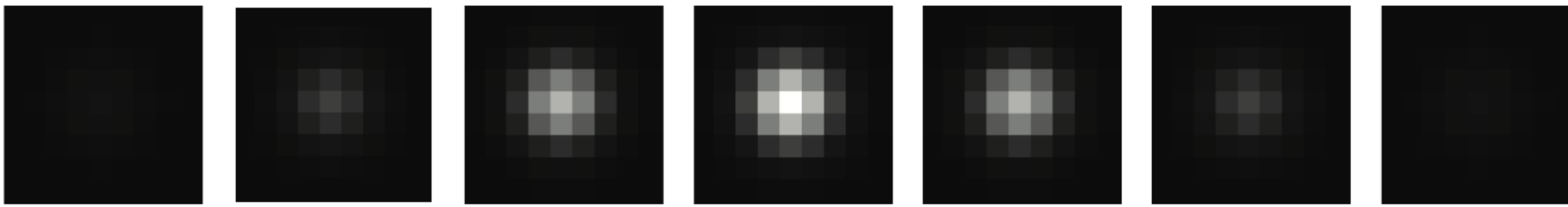
# Temporal Gaussian

$$g(x, y, t; \sigma_x, \sigma_t) = \frac{1}{(2\pi)^{3/2}\sigma_x^2\sigma_t} \exp - \frac{x^2 + y^2}{2\sigma_x^2} \exp - \frac{t^2}{2\sigma_t^2}$$

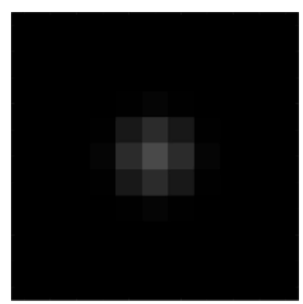



This filter keeps stationary things sharp, and blurs moving things.

# Spatio-temporal Gaussian



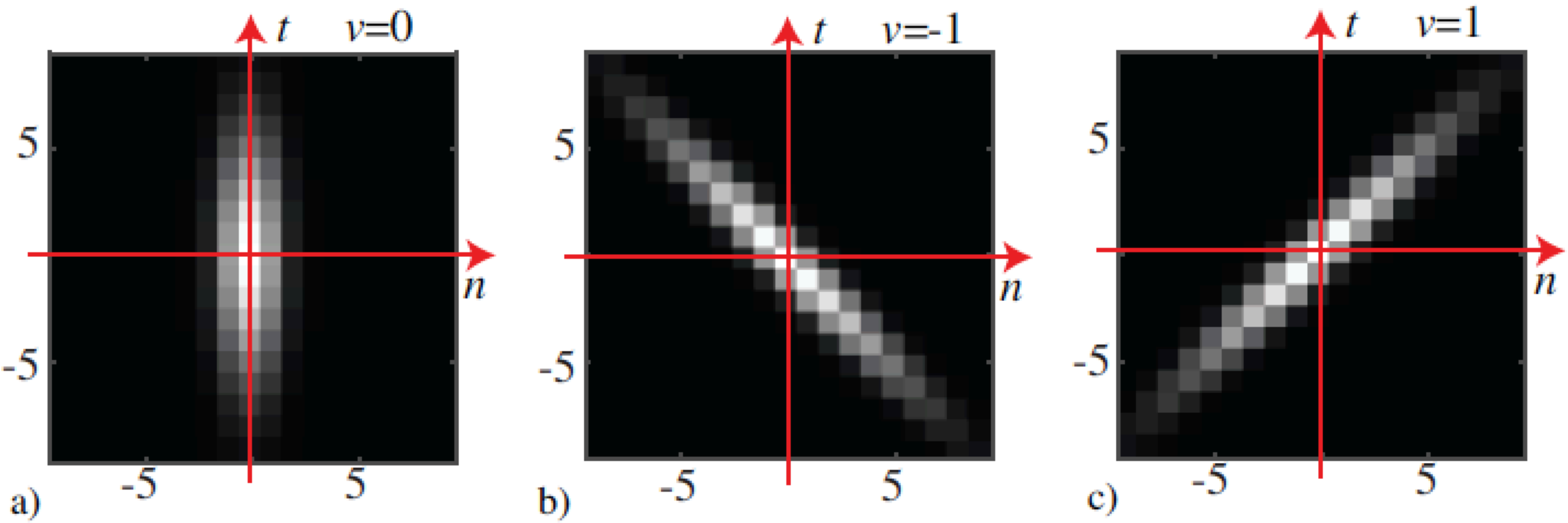
*t*=-3 *t*=-2 *t*=-1 *t*=0 *t*=1 *t*=2 *t*=3

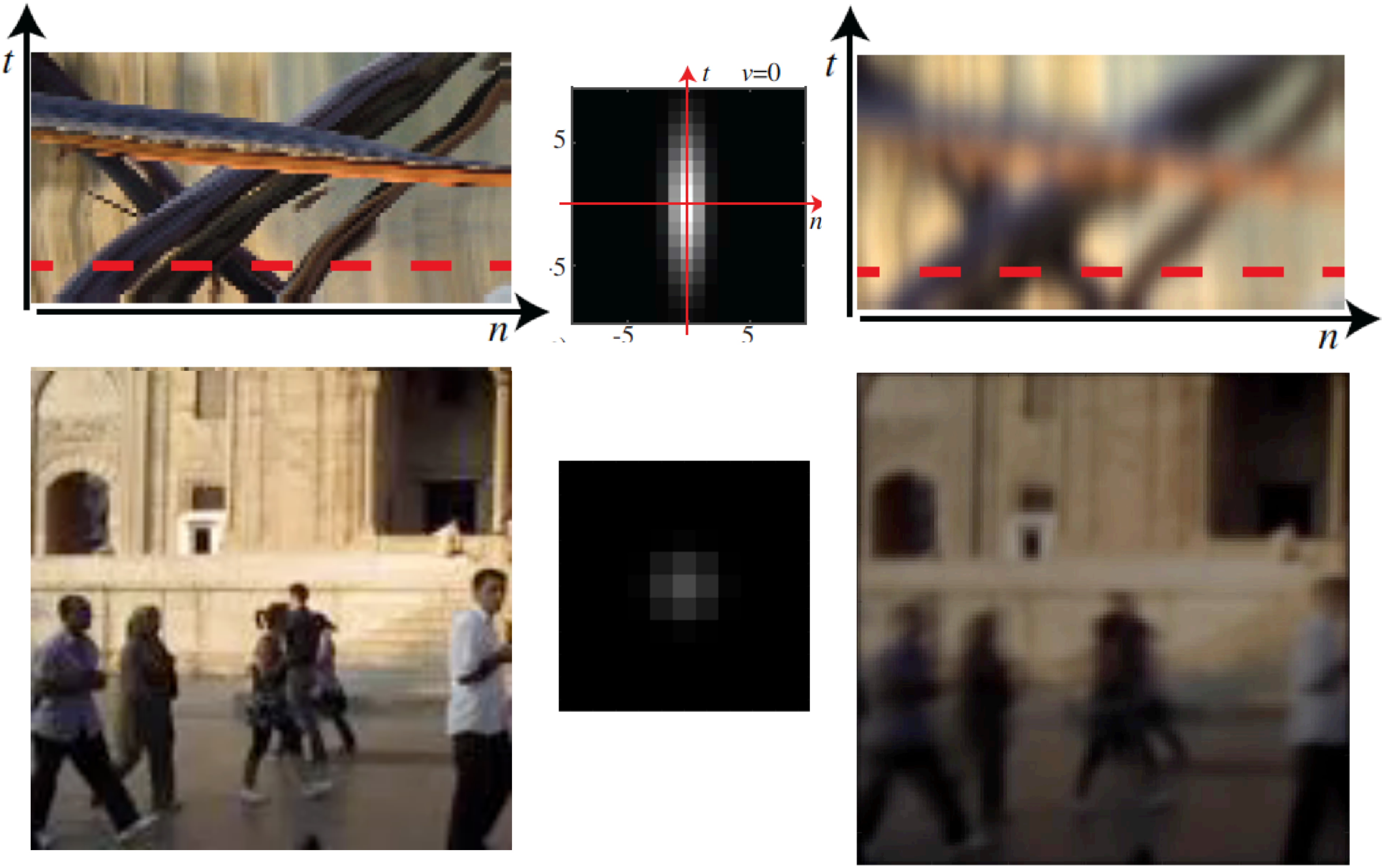


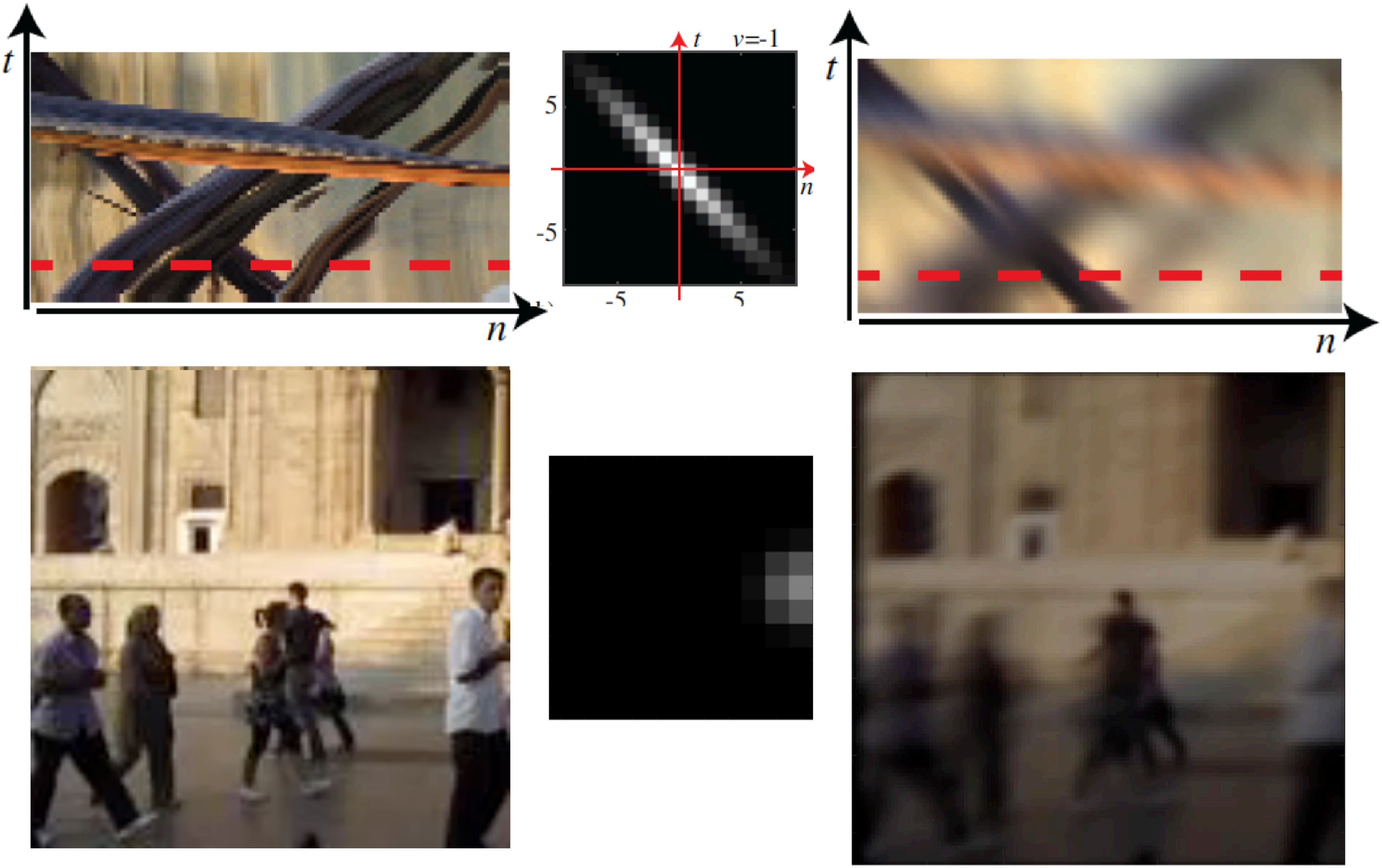
# Spatio-temporal Gaussian

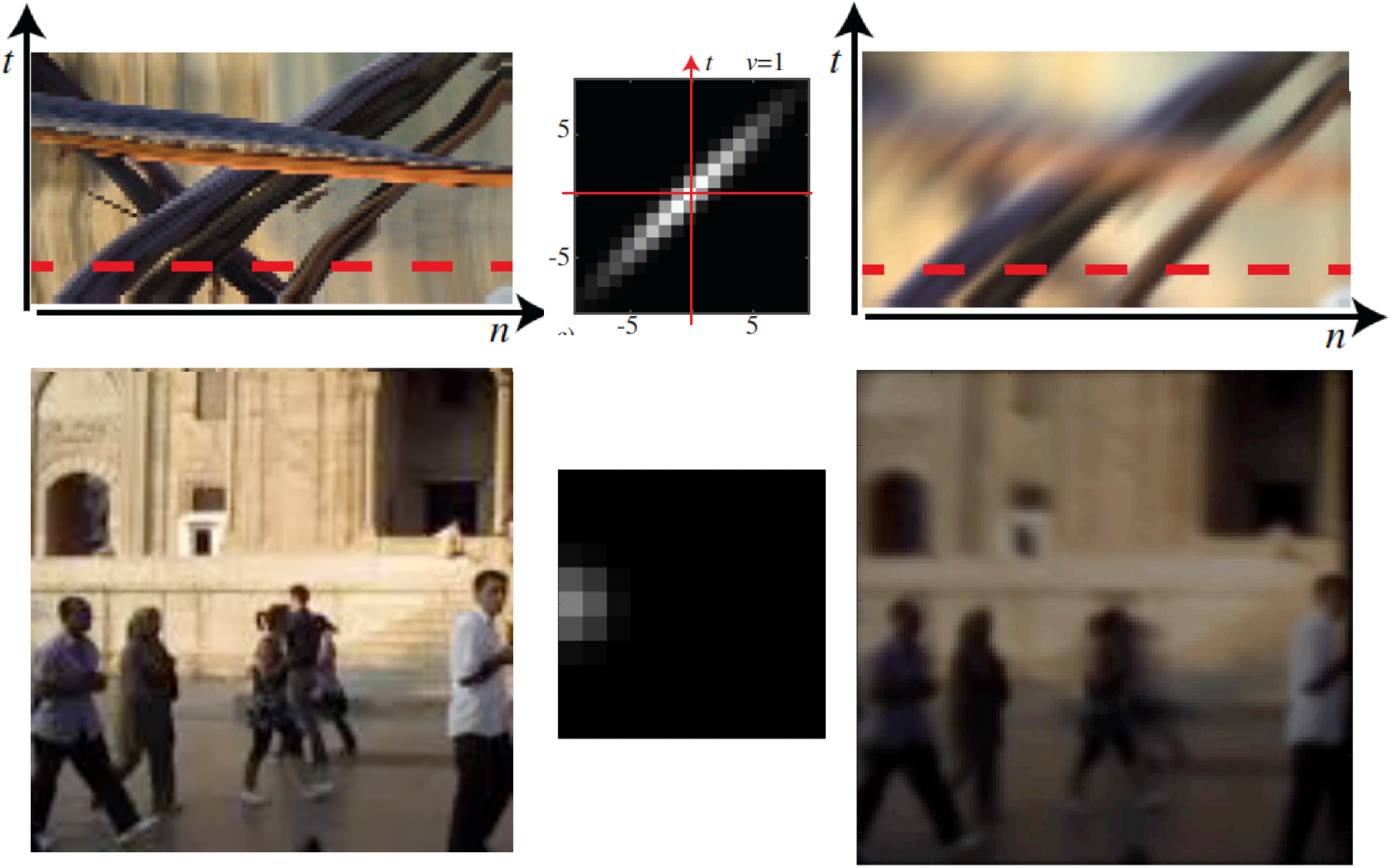
How could we create a filter that keeps sharp objects that move at some velocity **(vx , vy)** while blurring the rest?

$$g_{v_x,v_y}(x,y,t) = g(x - v_x t, y - v_y t, t)$$

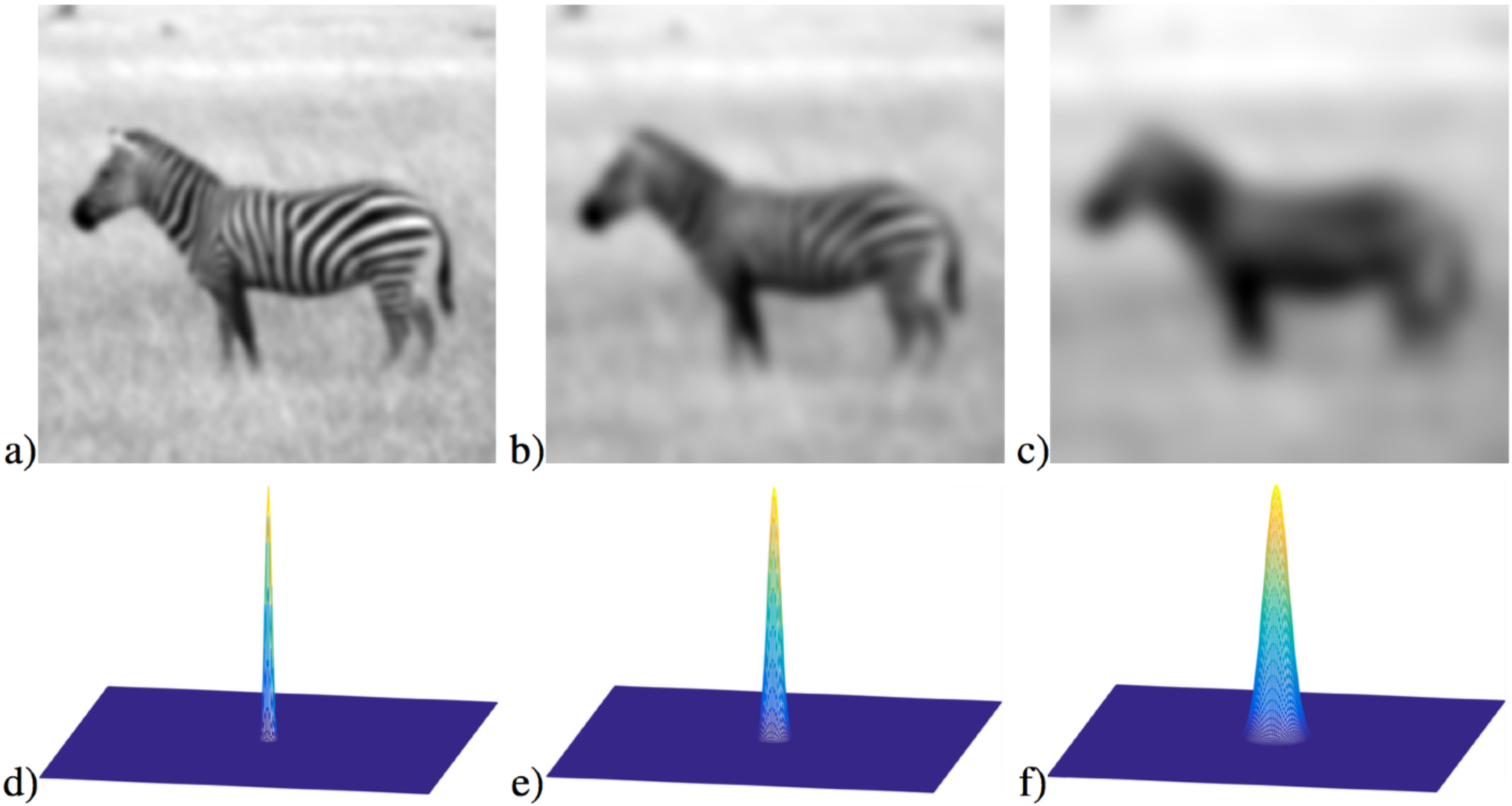



t
n
t
v=0
5
-5
n
-5
5
t
n


t
n
t
v=-1
5
-5
n
-5
5
t
n


t
n
t
v=1
5
-5
-5
5
t
n

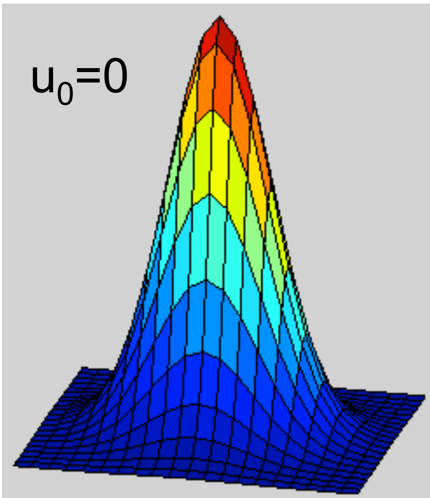
# (last class) Gaussians set scale

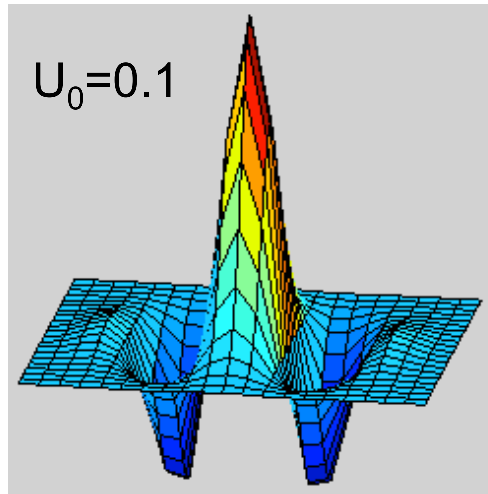


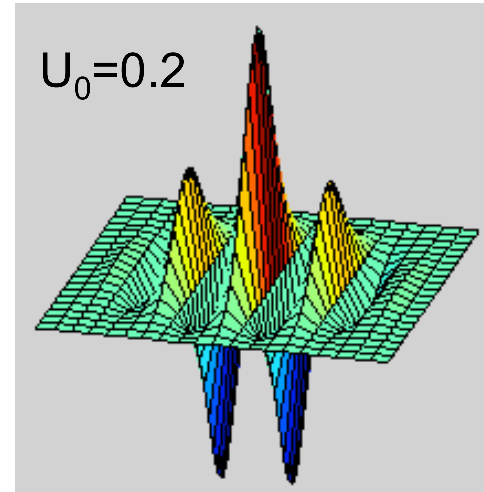
# Gabor wavelets

Good for both temporal and spatial filtering

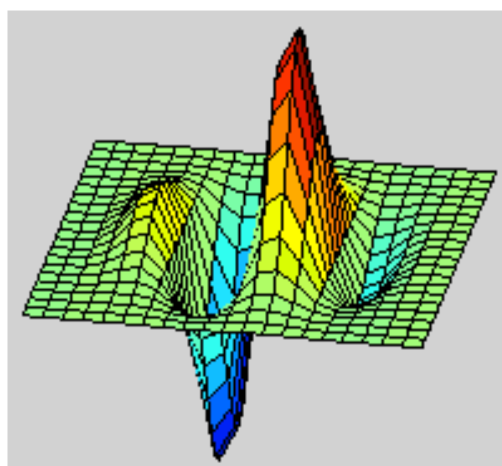
$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos\left(2\pi u_0 x\right)$$

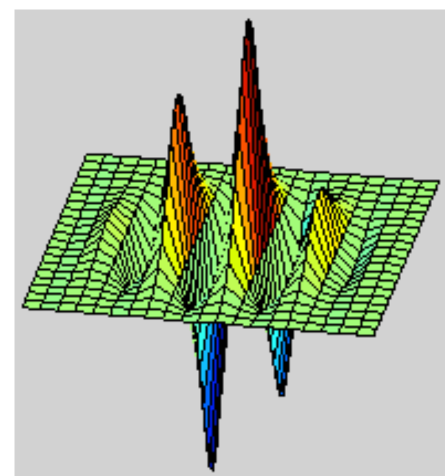


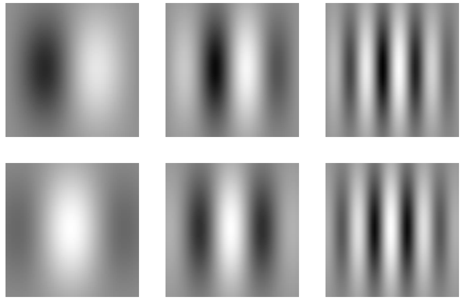




$$\psi_s(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \sin\left(2\pi u_0 x\right)$$

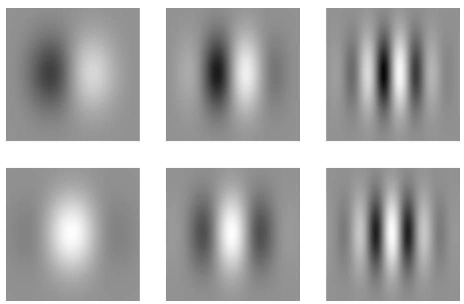




Gabor wavelets are like sinusoids, only they are localized, as enforced by the Gaussian multiplicative window.

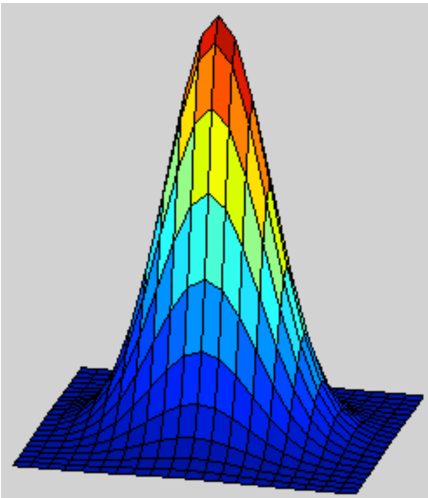


Gabor filters at different scales and spatial frequencies

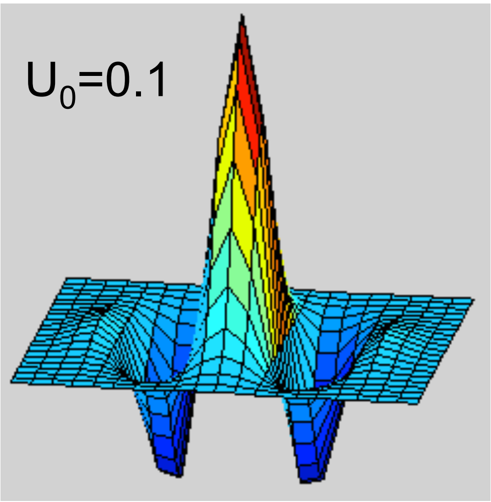


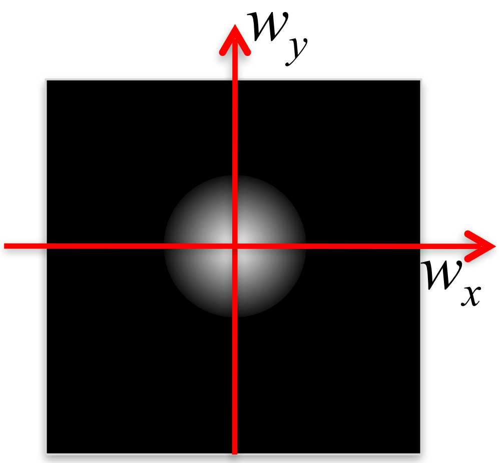
Top row shows anti-symmetric (or odd) filters; these are good for detecting odd-phase structures like edges.
Bottom row shows the symmetric (or even) filters, good for detecting line phase contours.

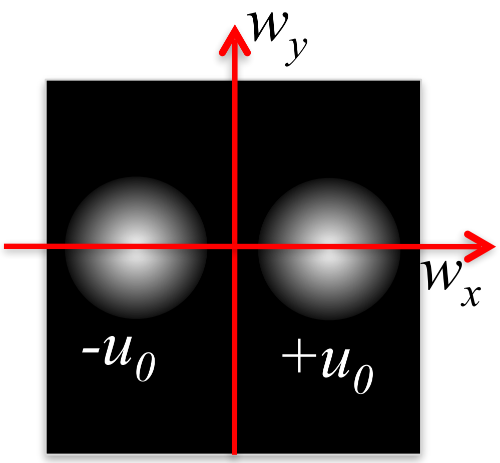
# Fourier transform of a Gabor wavelet

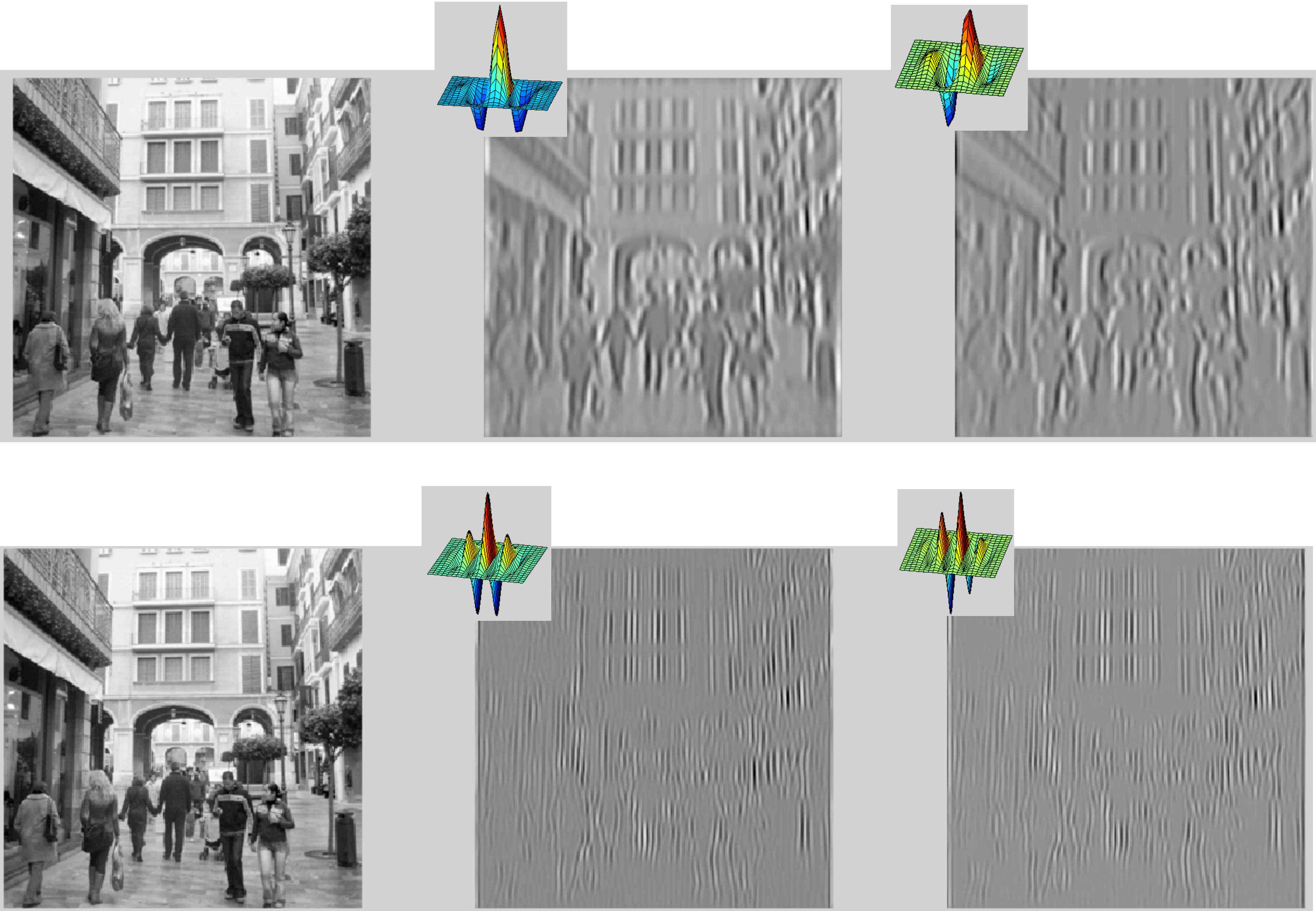


$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x)$$



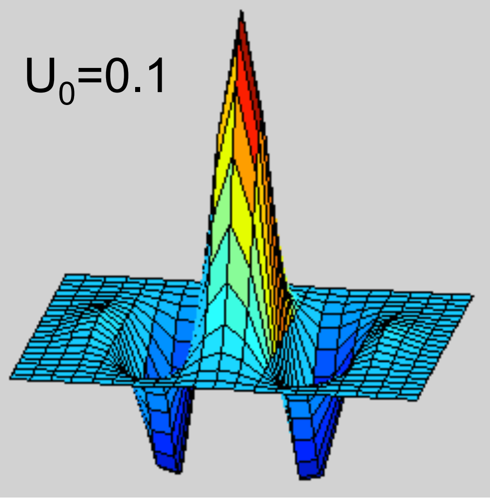



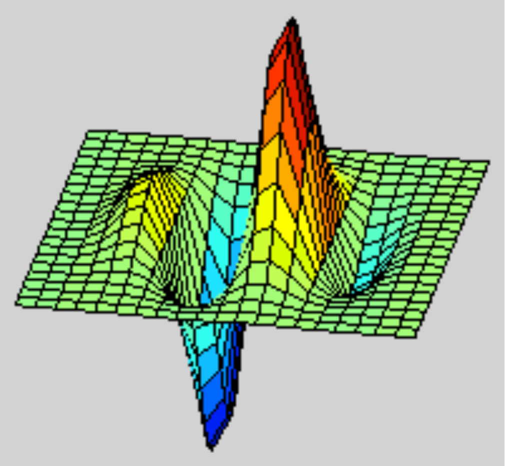


# Quadrature pair

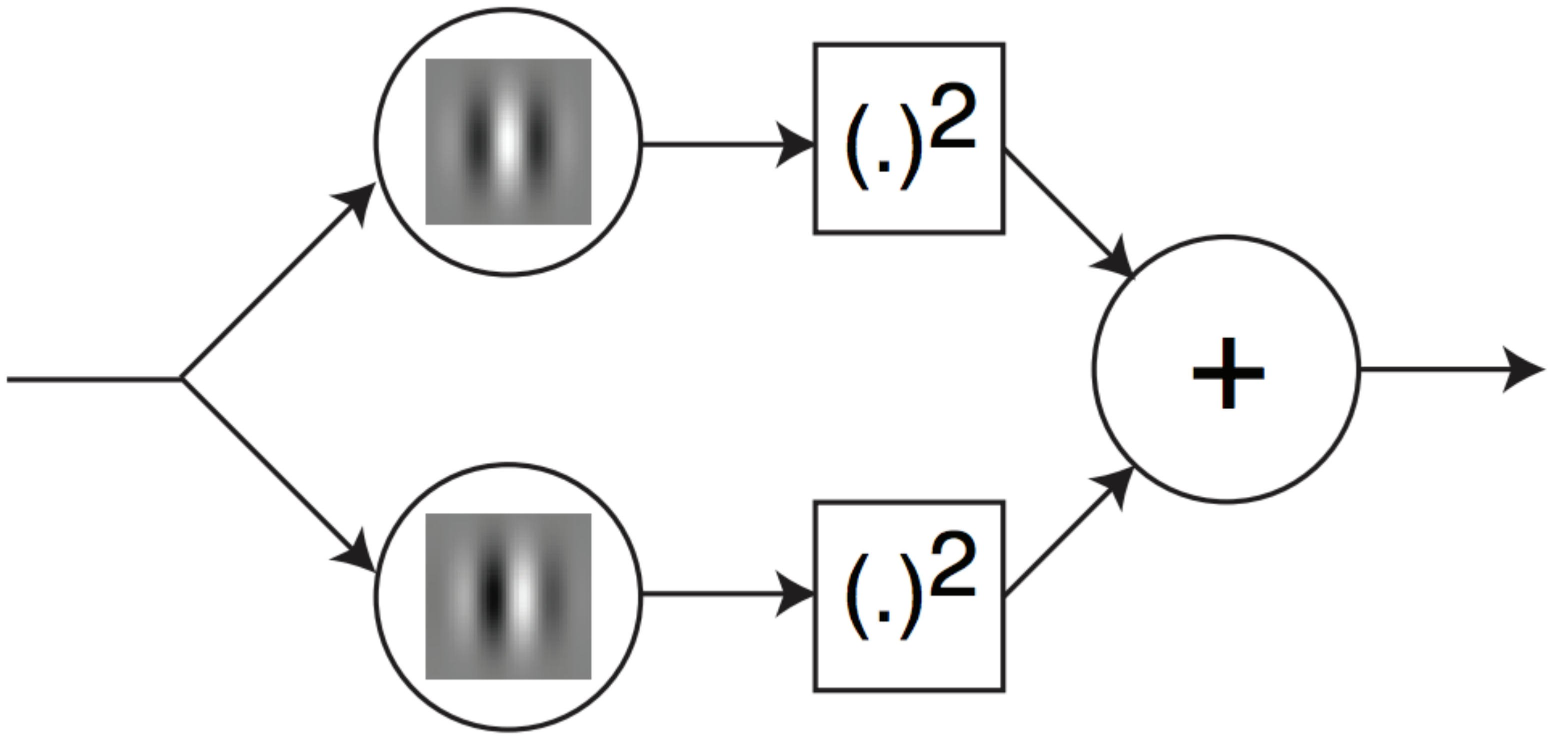
$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x)$$




$$\psi_s(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \sin(2\pi u_0 x)$$

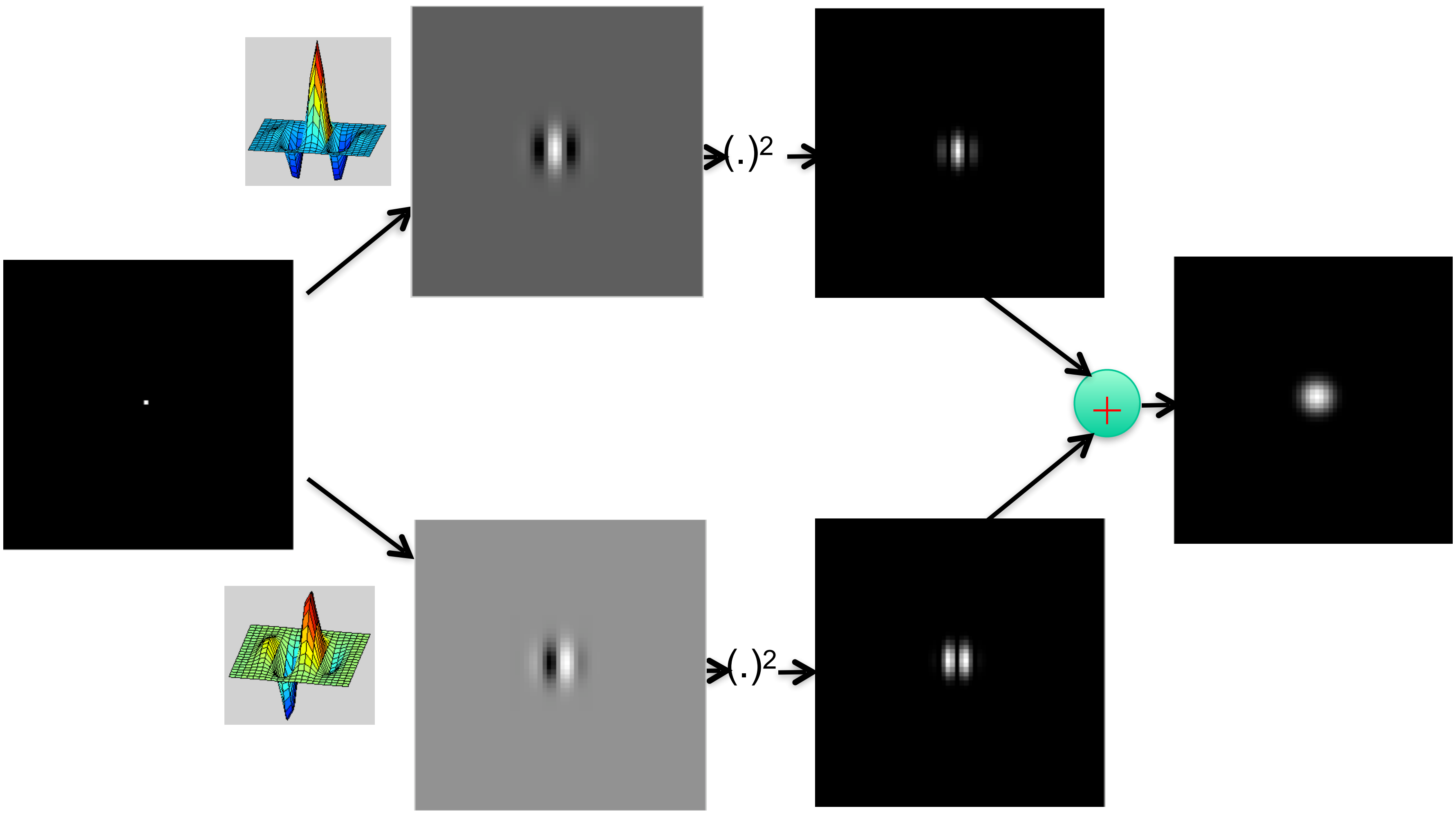


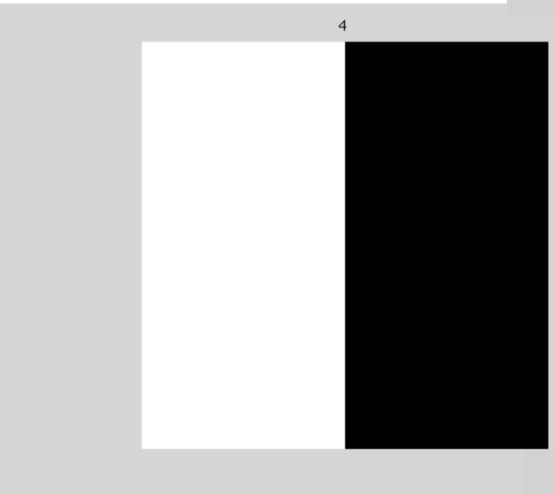
# “oriented energy” from a quadrature pair



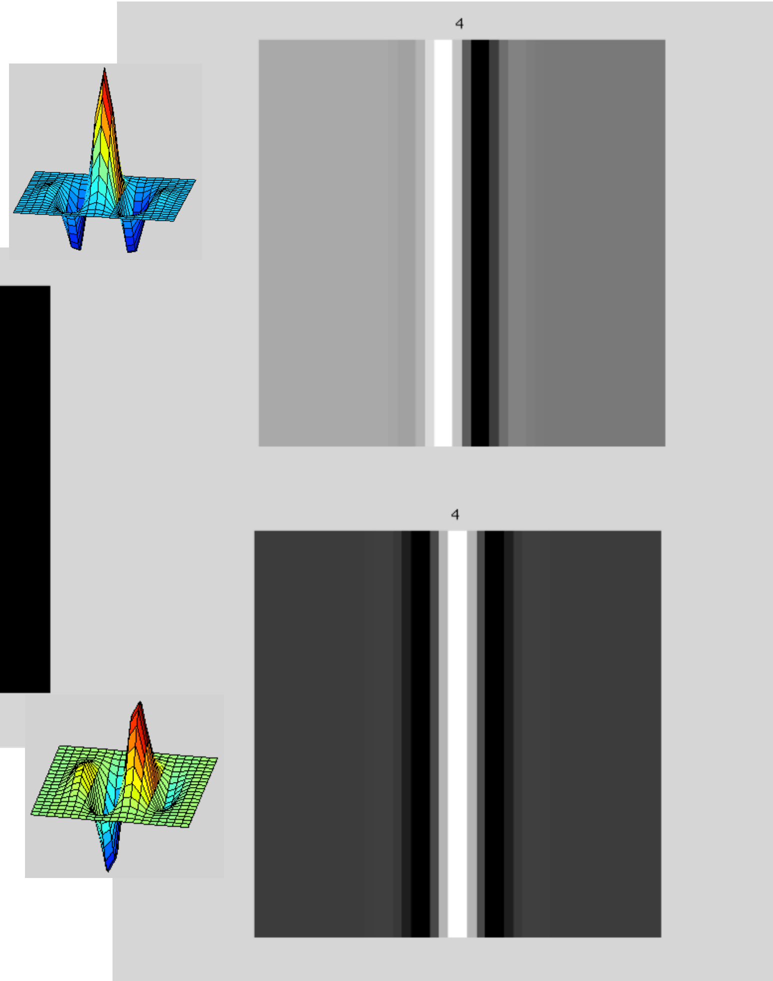
# Quadrature filter pairs

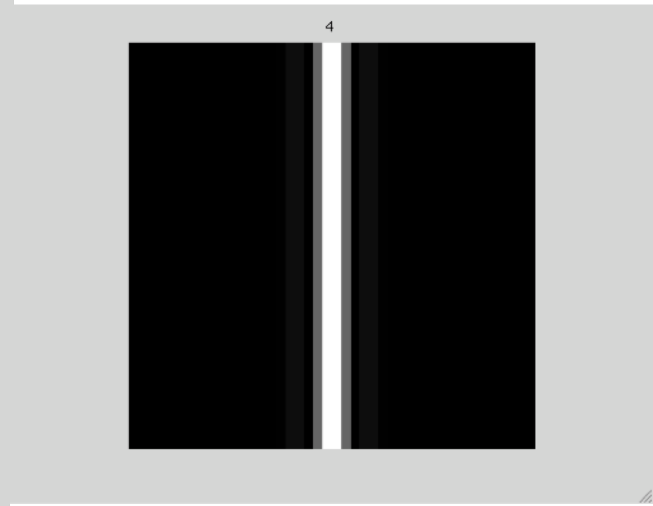
A quadrature filter is a complex filter whose real part is related to its imaginary part via a Hilbert transform along a particular axis through origin of the frequency domain.



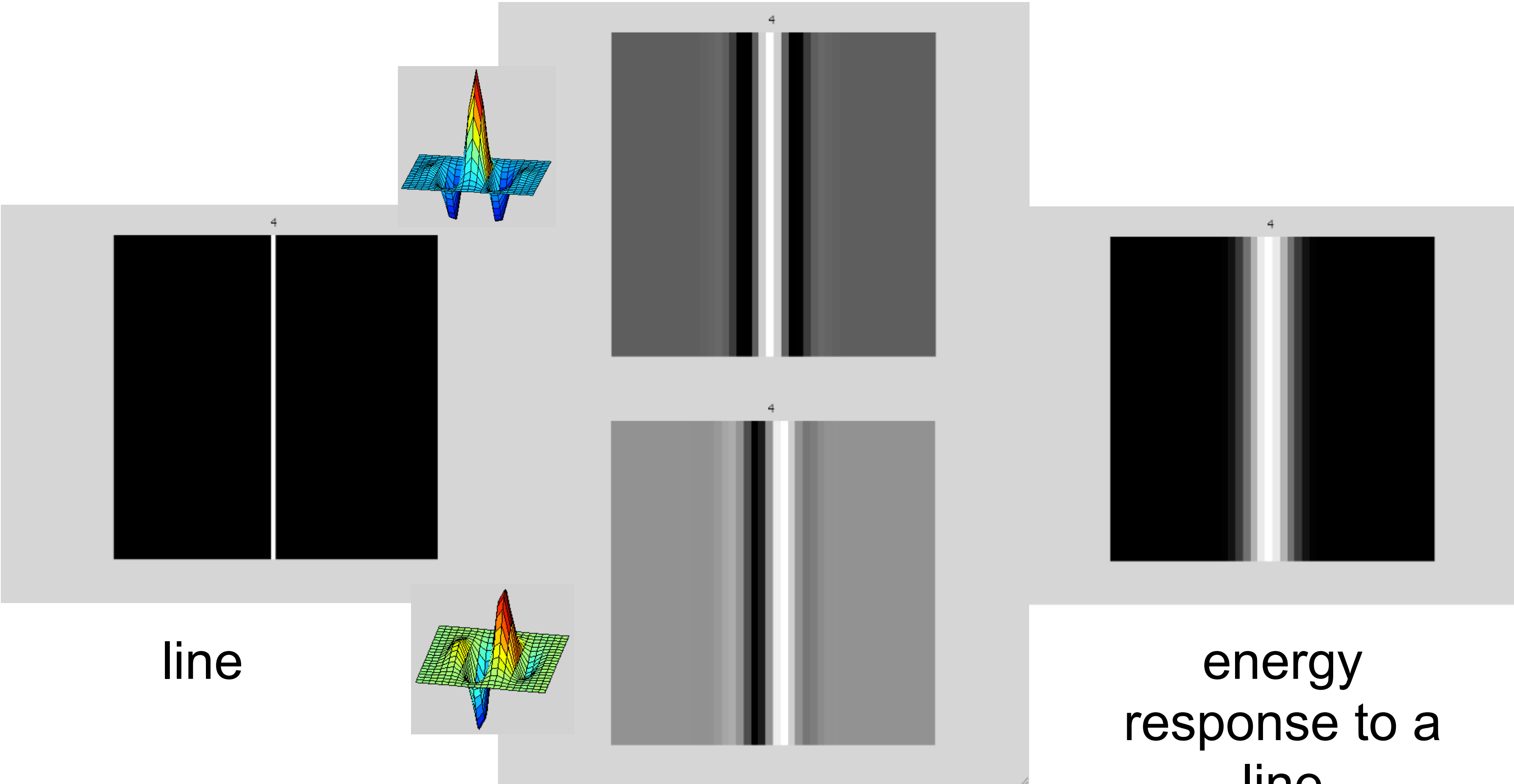


edge





energy response to an edge

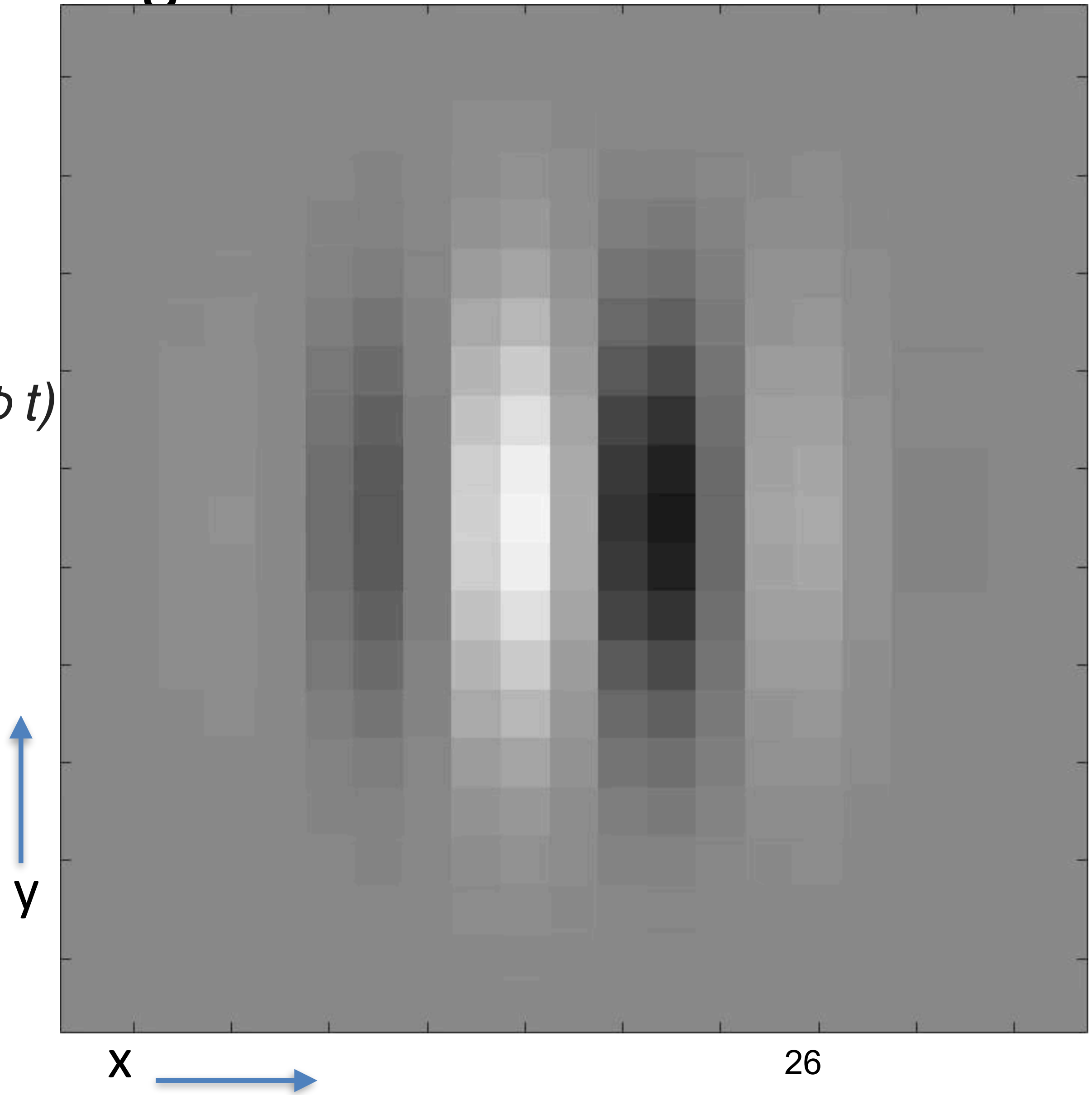


line

energy response to a line

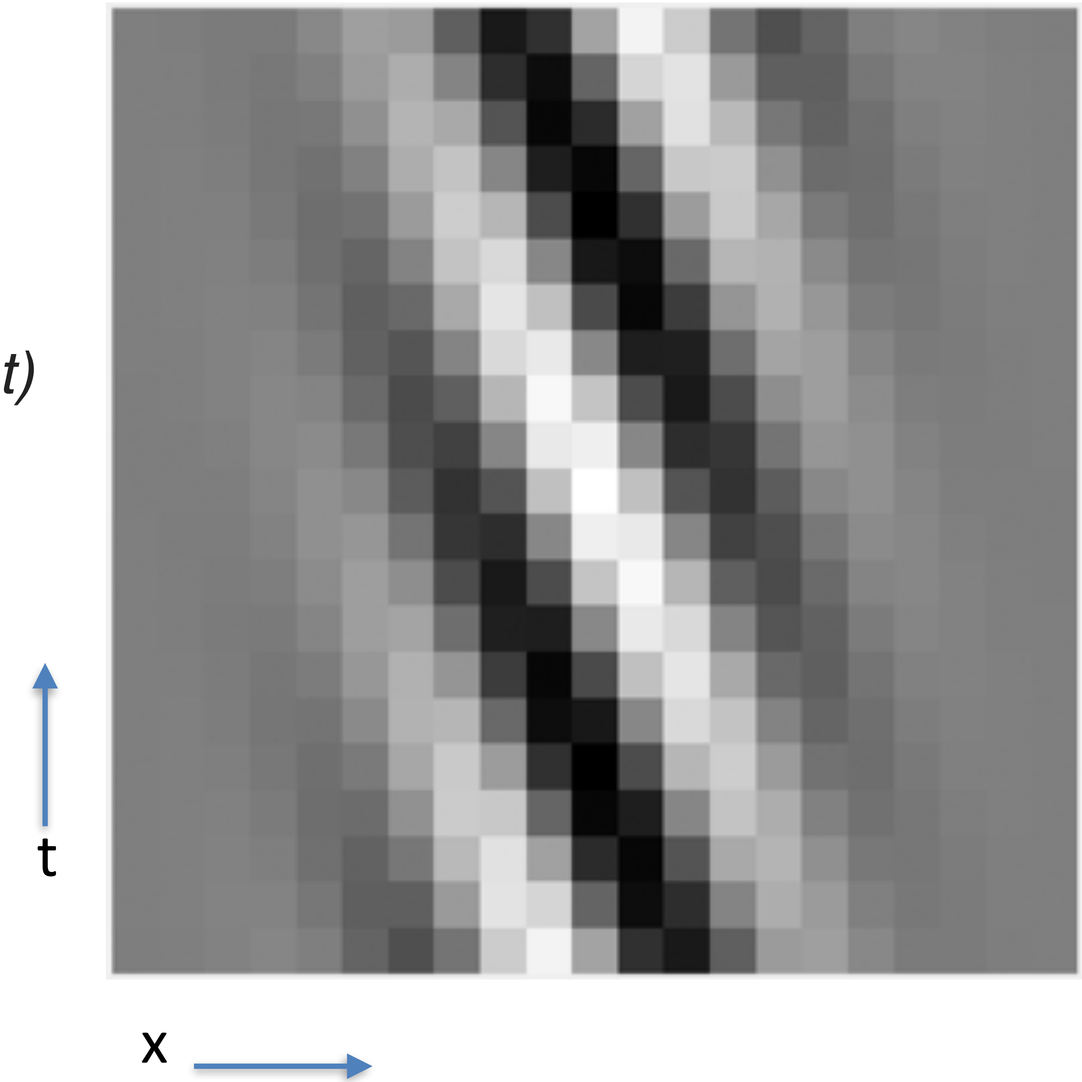
# Using phase changes of local Gabor filters to analyze or generate motion

$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x + \phi\, t)$$

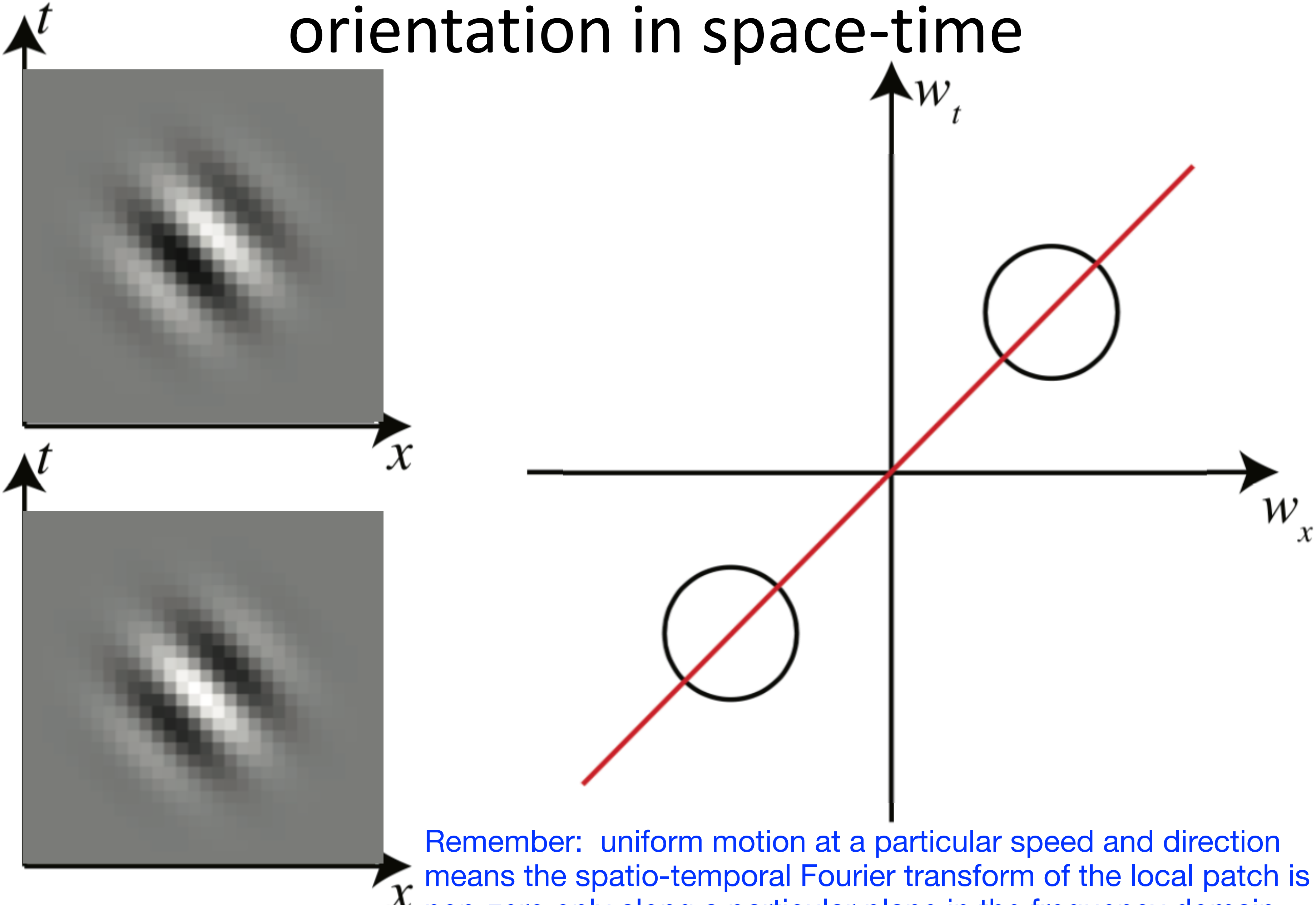


# Space-time plot of the a slice through the patio-temporal filter of the previous slide

$$\psi_c(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \cos(2\pi u_0 x + \phi\, t)$$



# Gabor filters for analyzing motion as orientation in space-time




Remember: uniform motion at a particular speed and direction means the spatio-temporal Fourier transform of the local patch is non-zero only along a particular plane in the frequency domain.

# Gabor filters for analyzing motion

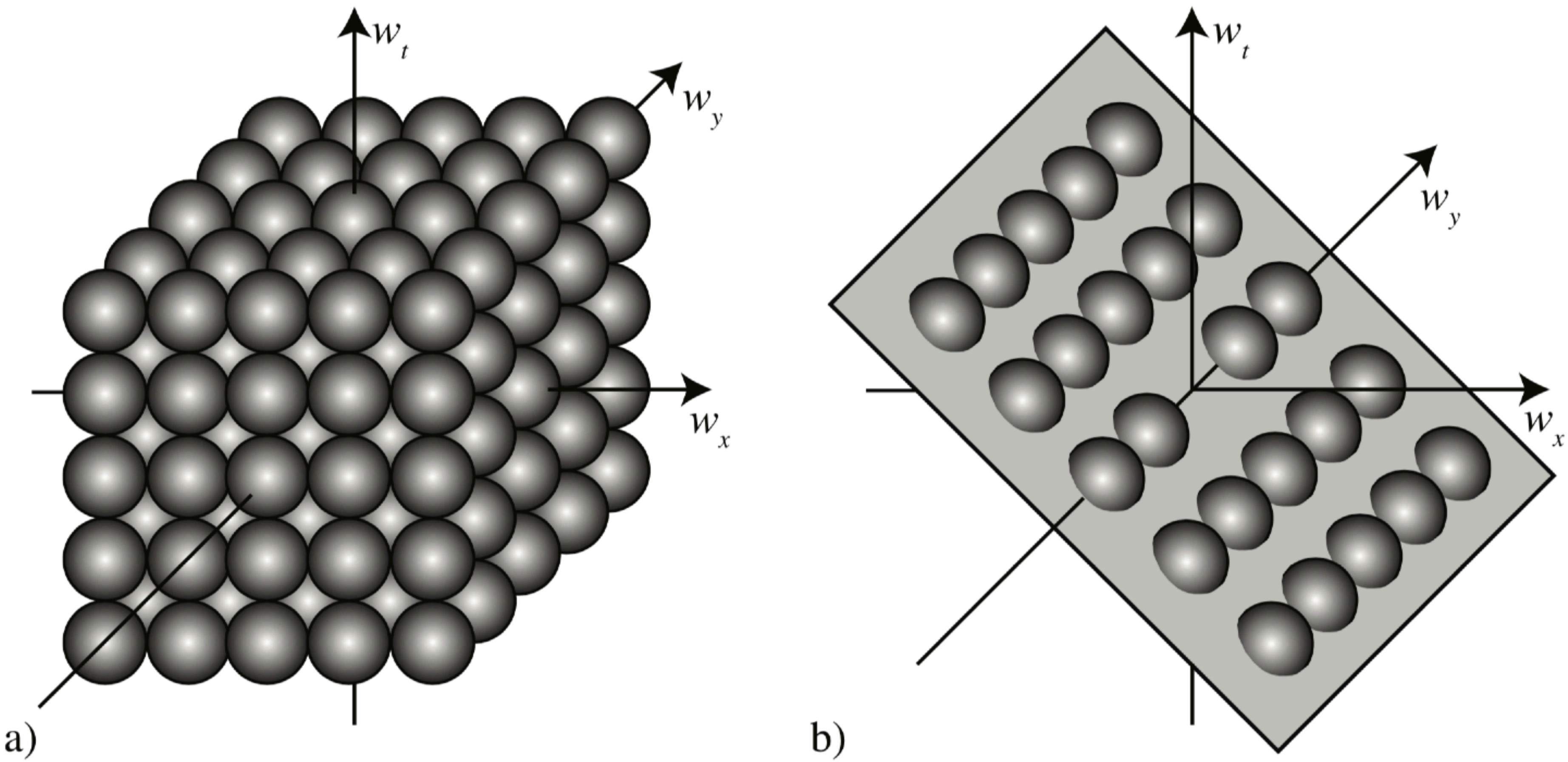



Figure 1.9: a) Space-time Gabor filters tiles. b) Set of Gabor filters selective to a particular velocity.

# Motion without movement

SIGGRAPH '91 Las Vegas, 28 July-2 August 1991

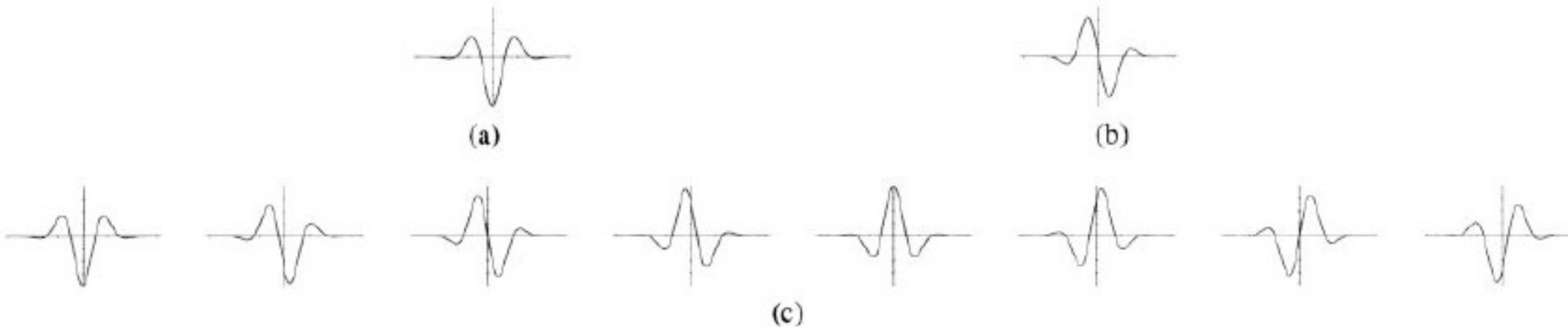


Figure 1: 1-d cross-sections of filters. (a) Even phase ($G_2$). (b) Odd phase ($H_2$). (c) Filters modulated in phase according to Eq. (1). Note the apparent rightward motion of the filter ripples.

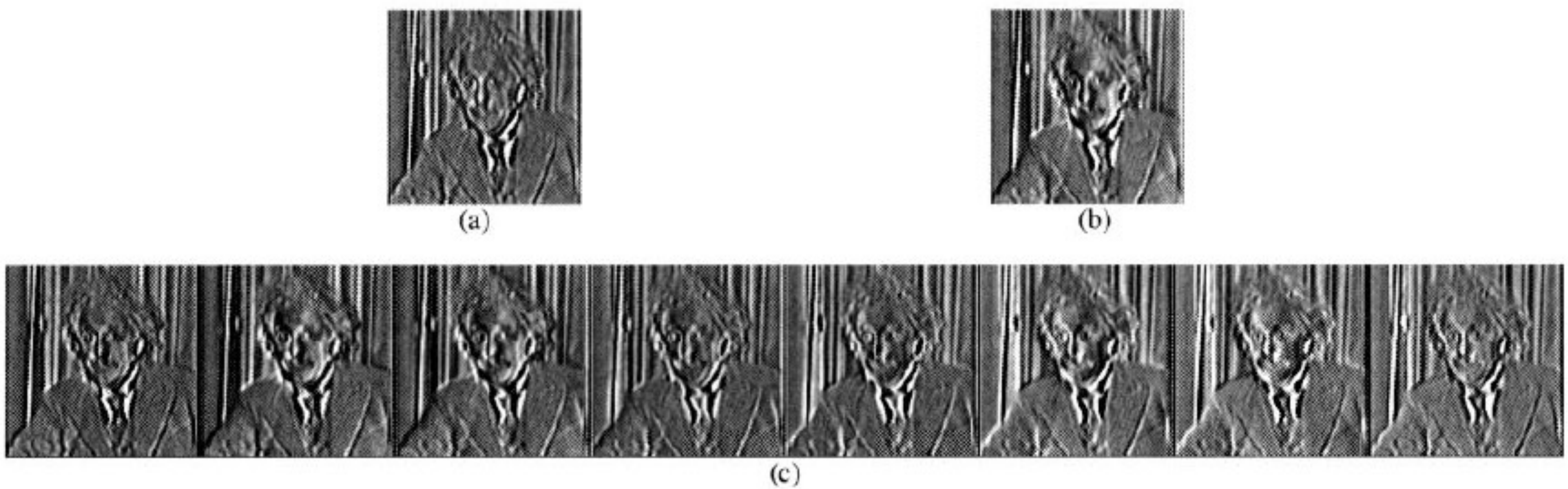
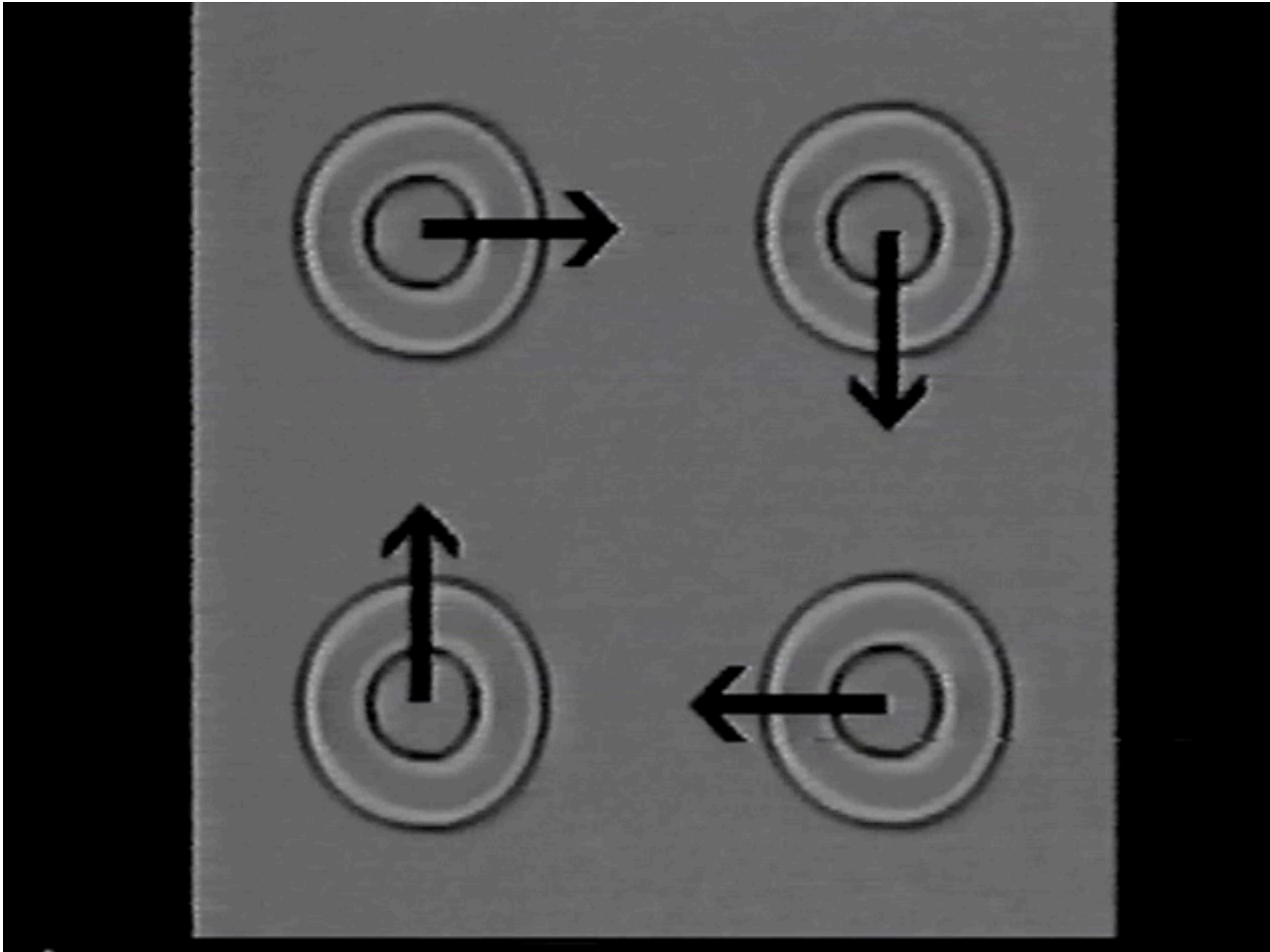


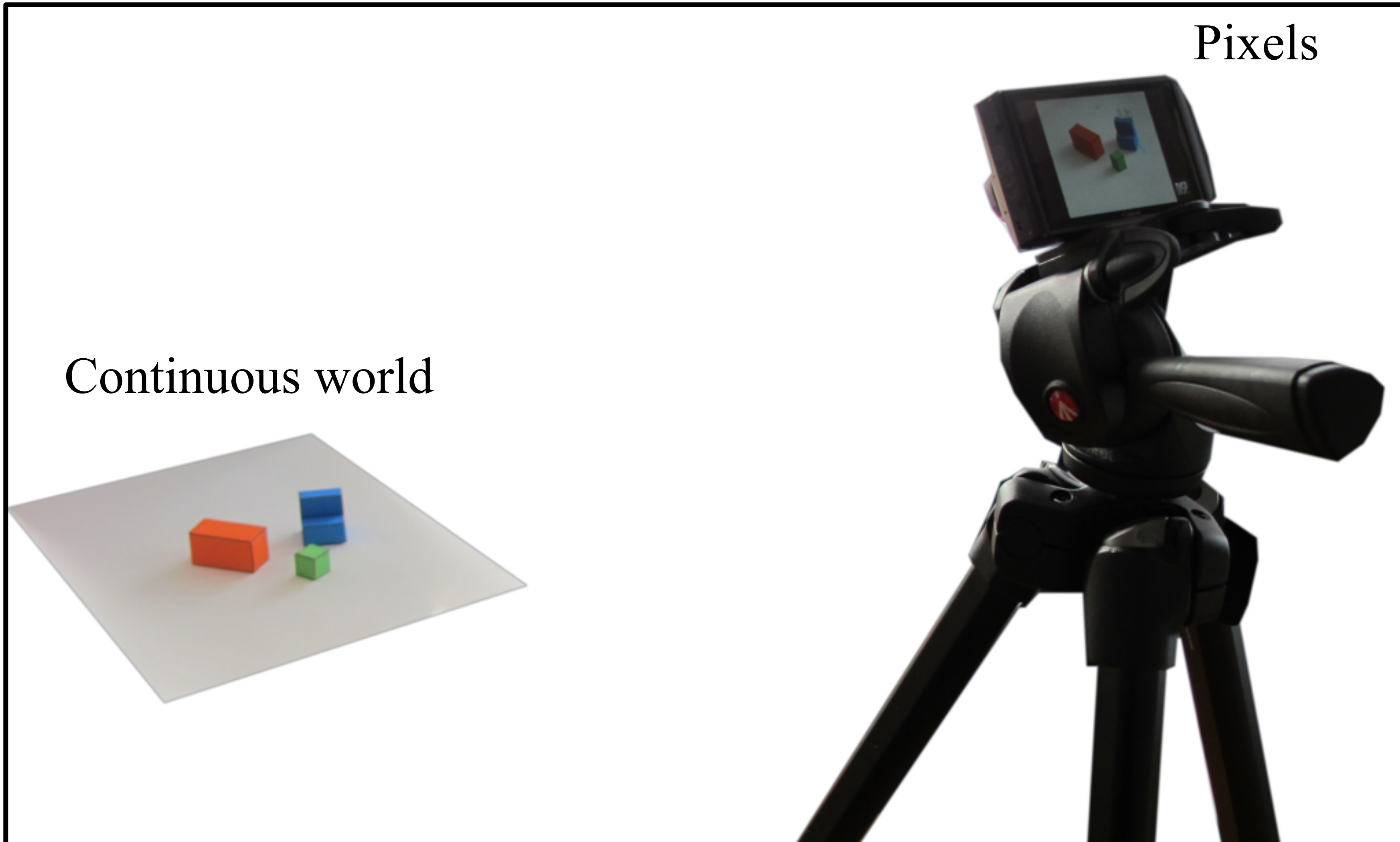
Figure 2: (a) and (b): $G_2$ and $H_2$ filters were applied to an image of Einstein. (c) Images modulated as in Eq. (1). When viewed as a temporal sequence, this generates the perception of rightward motion, yet image remains stationary.

# Motion without movement

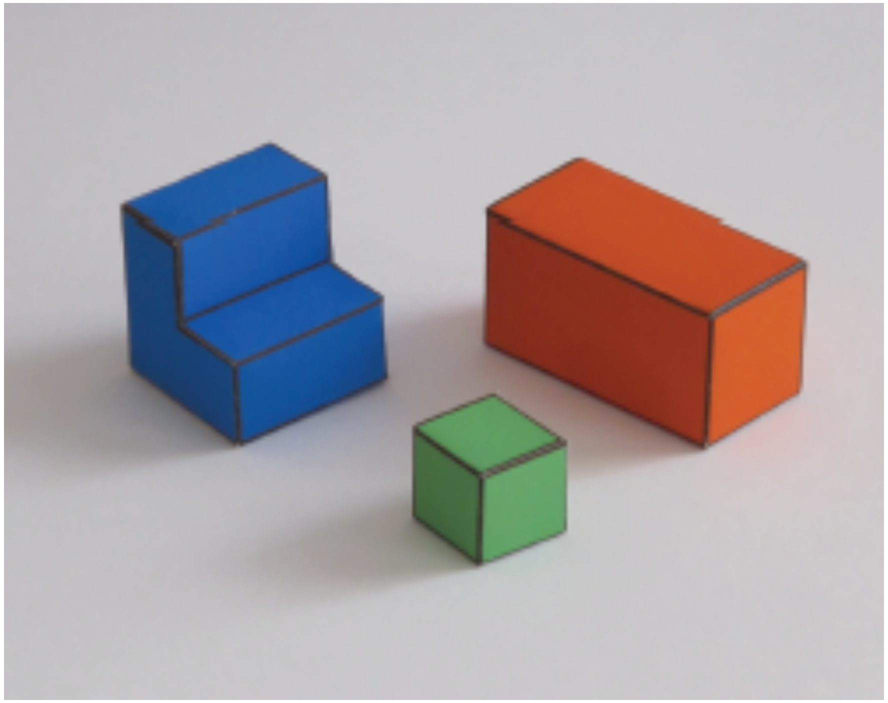


# Sampling

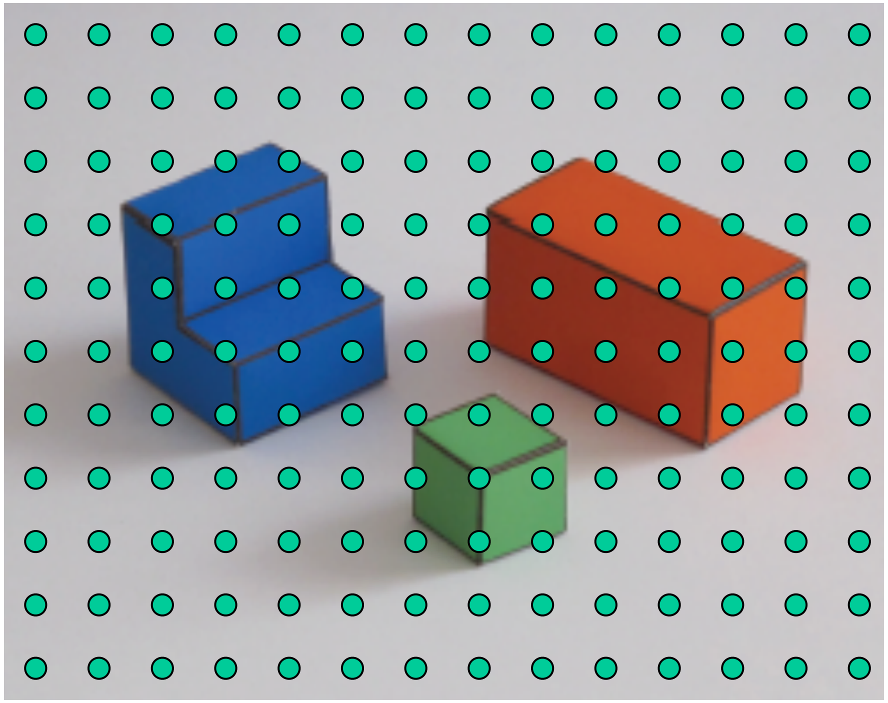
# Sampling



# Sampling



# Sampling

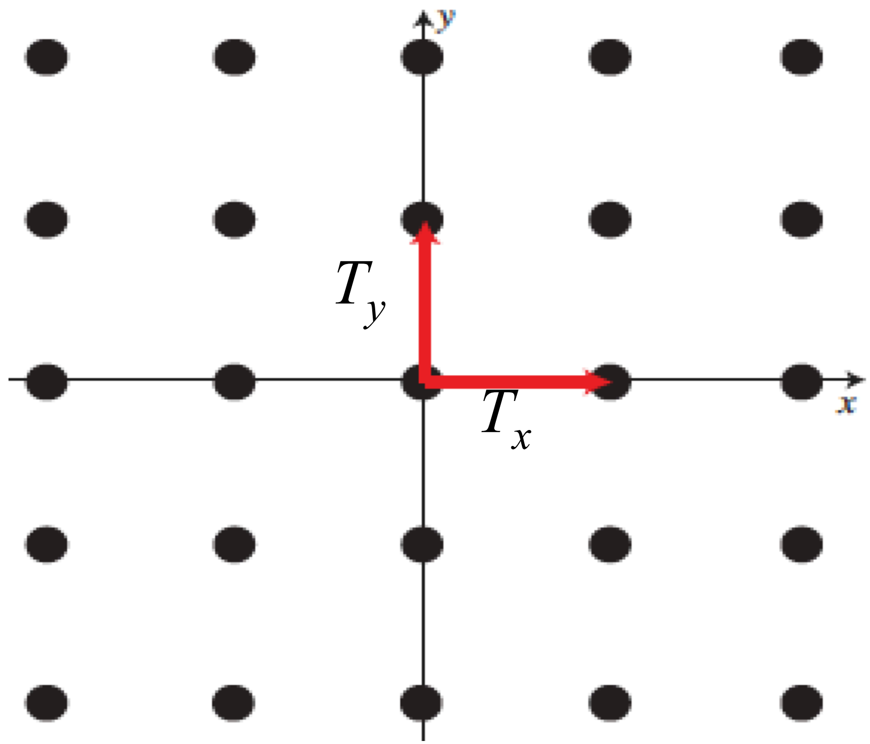


# Sampling

Continuous image $f(x, y)$

We can sample it using a rectangular grid as

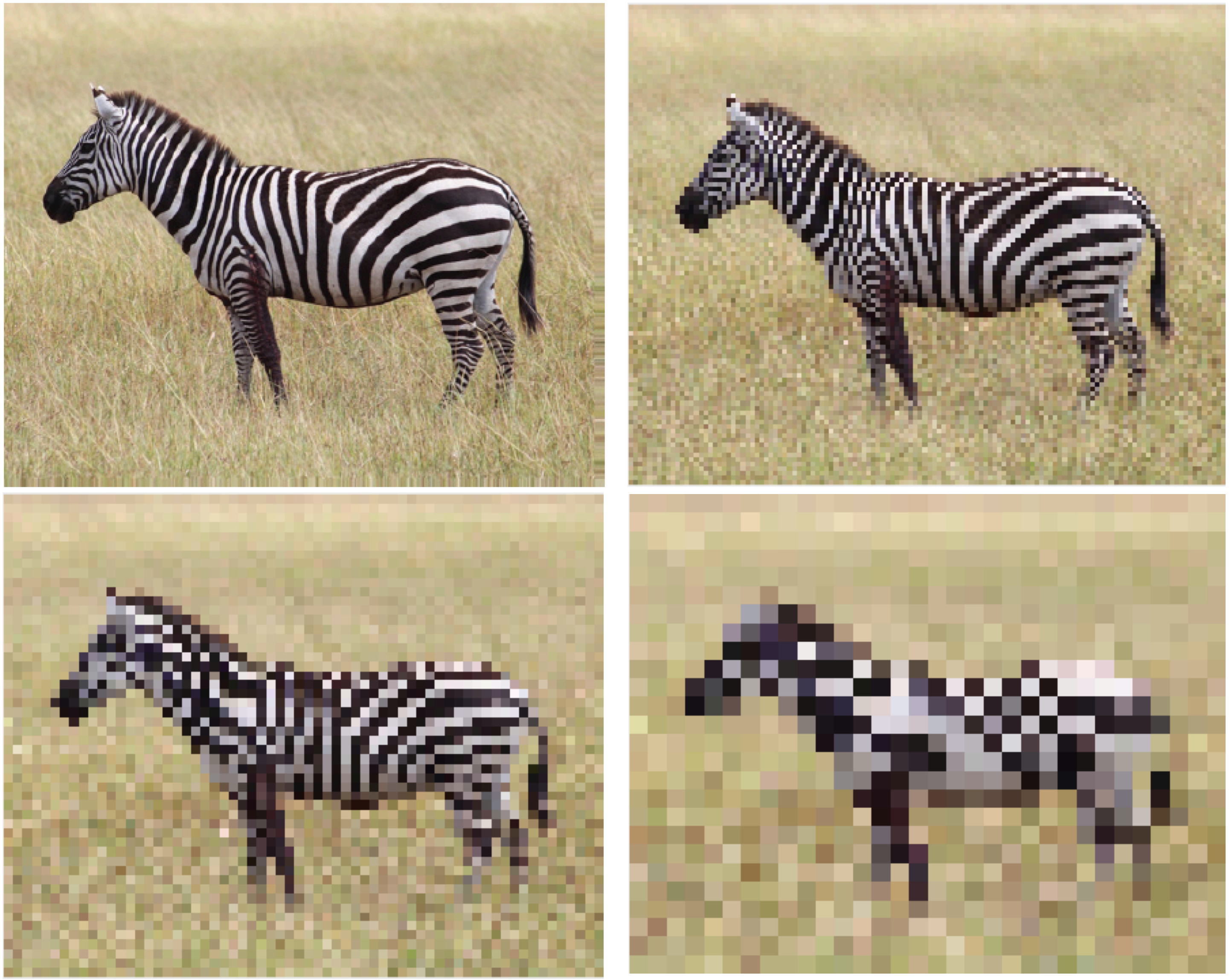
$$f[n,m] = f\left(nT_x, mT_y\right)$$



# Aliasing

Let's start with this continuous image (it is not really continuous…)

# Aliasing



103×128

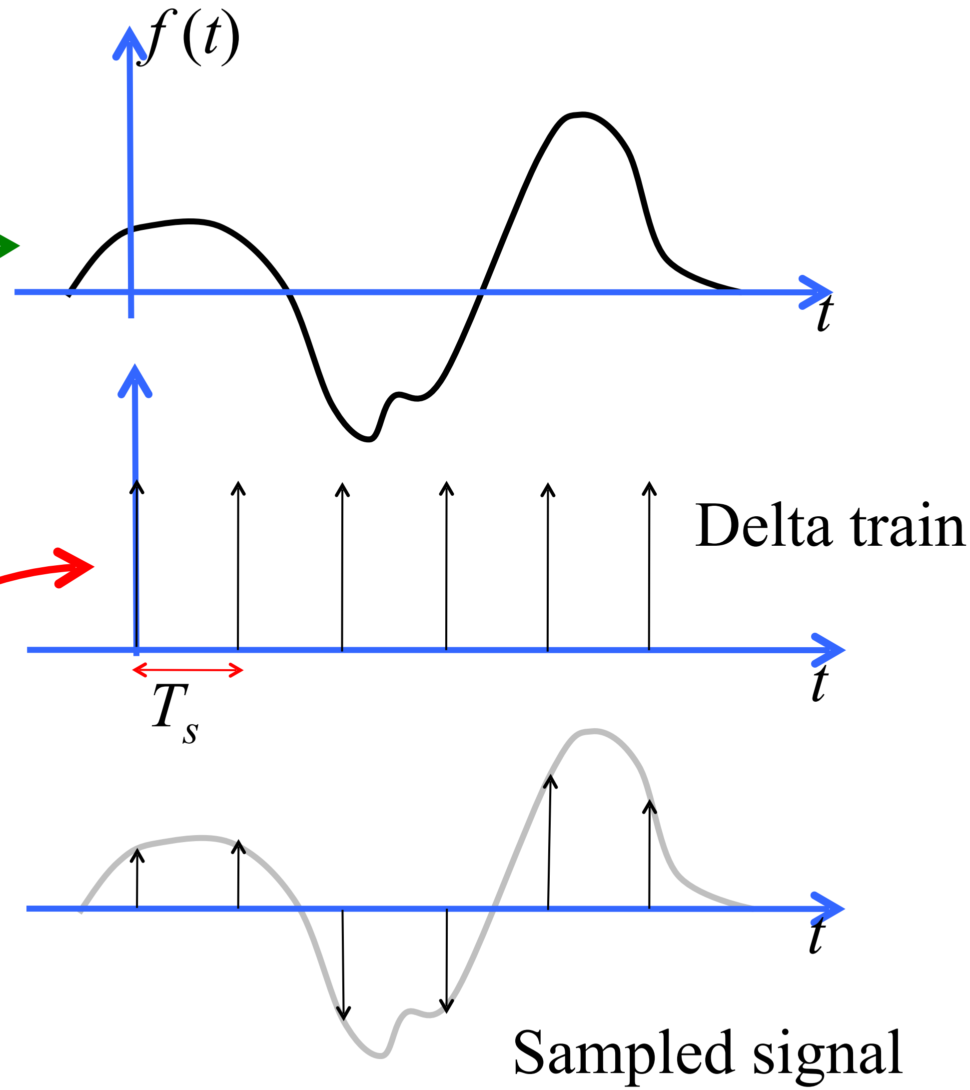
52×64

26×32

# Modeling the sampling process

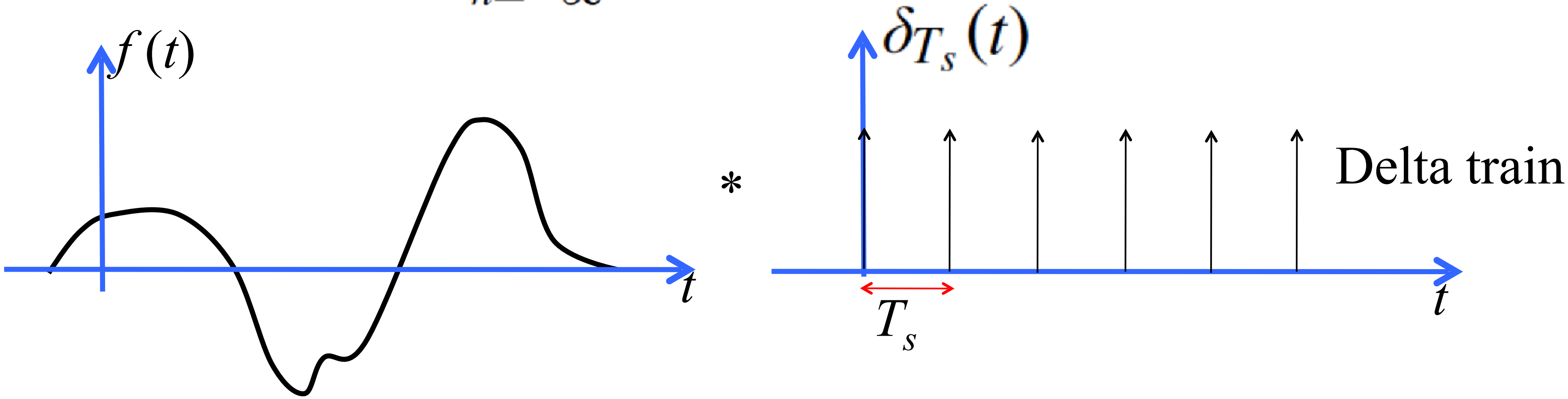
$$f[n] = f(nT_s)$$

$$\hat{f}(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



# Modeling the sampling process

$$\hat{f}(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = f(t)\delta_{T_s}(t)$$

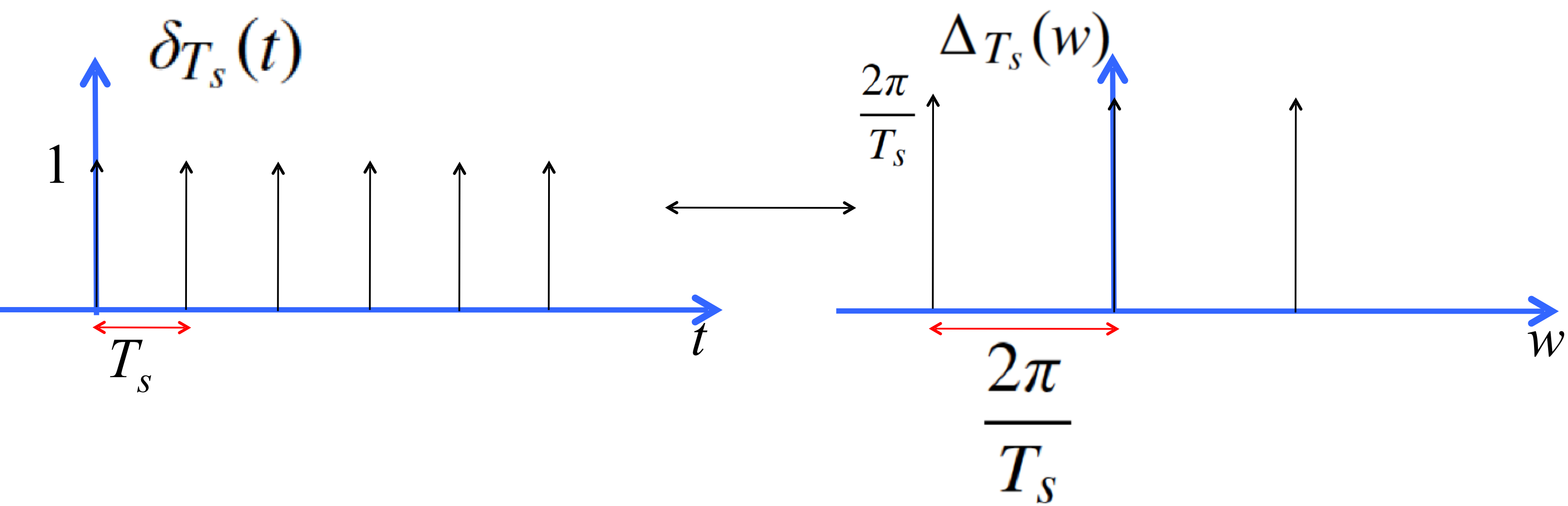



The Fourier transform is a convolution…

Interesting property of the delta train: the Fourier transform of a delta train of period T is another delta train with period 2π/T

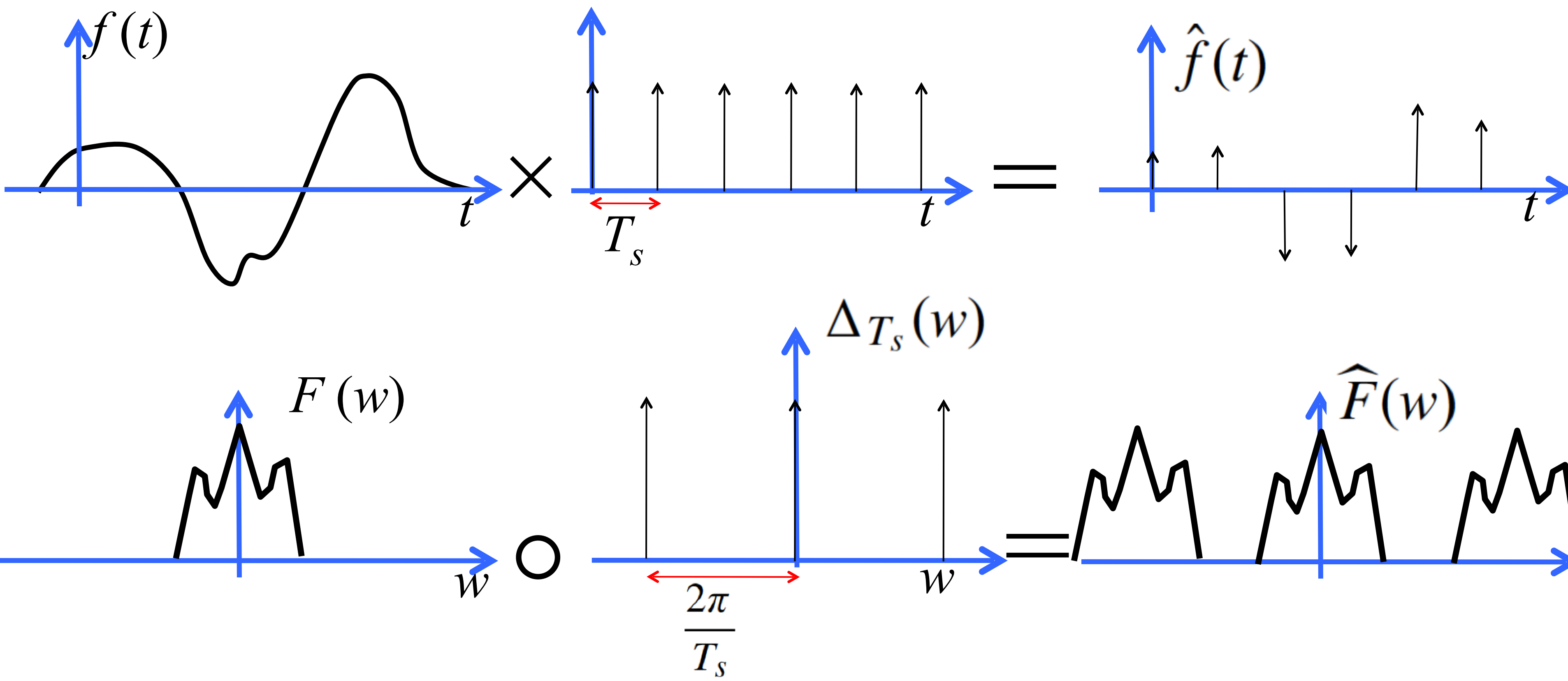
# Modeling the sampling process

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \longleftrightarrow \Delta_{T_s}(w) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(w - k\frac{2\pi}{T_s}\right)$$

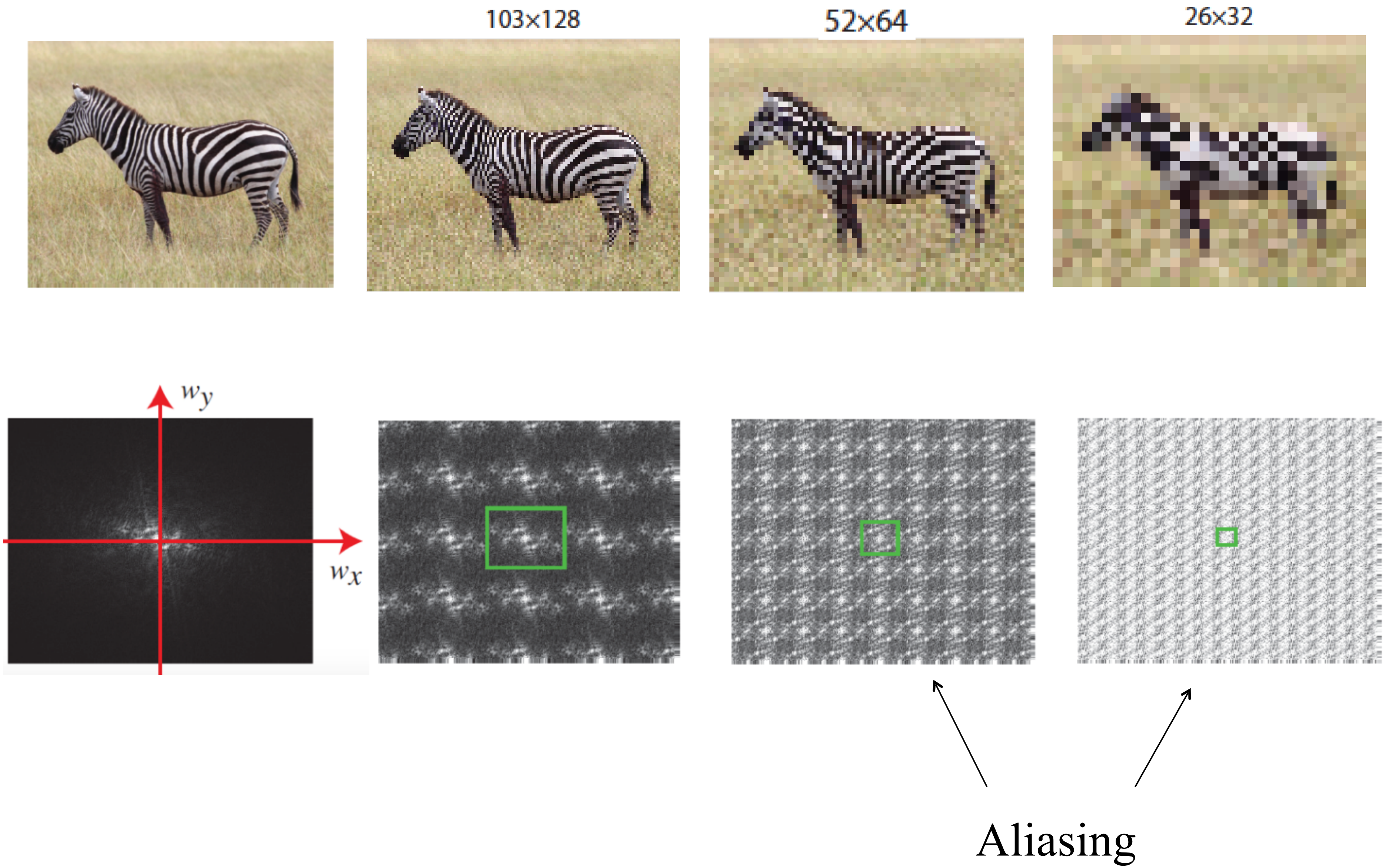



Interesting property of the delta train: the Fourier transform of a delta train of period T is another delta train with period 2π/T.

# Modeling the sampling process

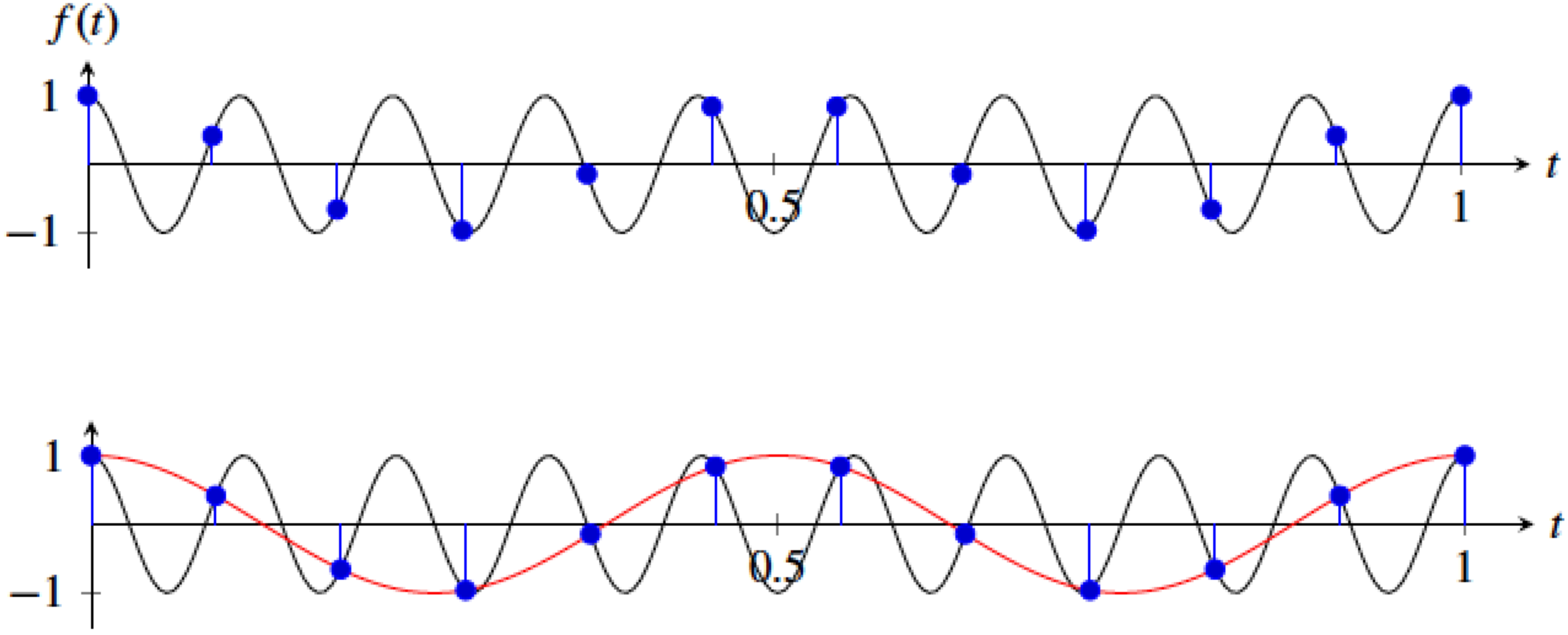



What happens when the repetitions overlap?



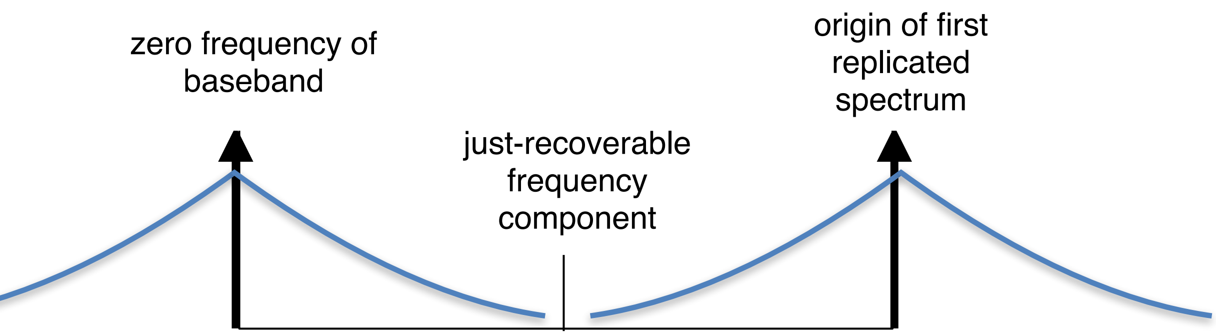
103×128
52×64
26×32
$w_y$
$w_x$
Aliasing

# Aliasing

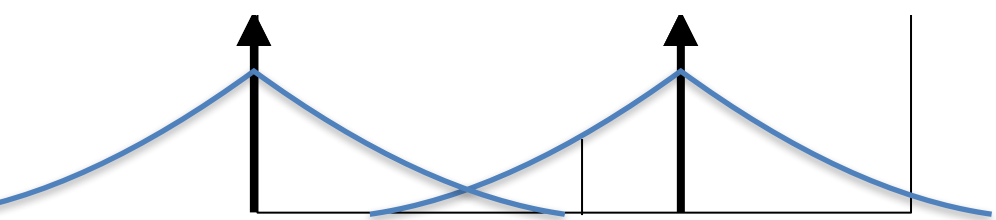



Both waves fit the same samples. Aliasing consists in "perceiving" the red wave when the actual input was the blue wave.

# aliasing



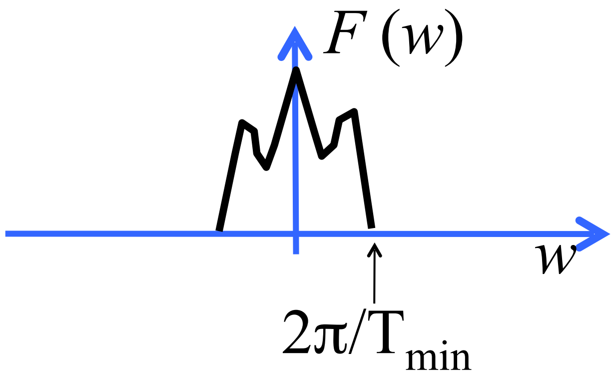

**no aliasing**



**aliasing**

# Sampling theorem

The sampling theorem (also known as Nyquist theorem) states that for a signal to be perfectly reconstructed from it samples, the sampling period $T_s$ has to be $T_s > T_{min}/2$ where $T_{min}$ is the period of the highest frequency present in the input signal.



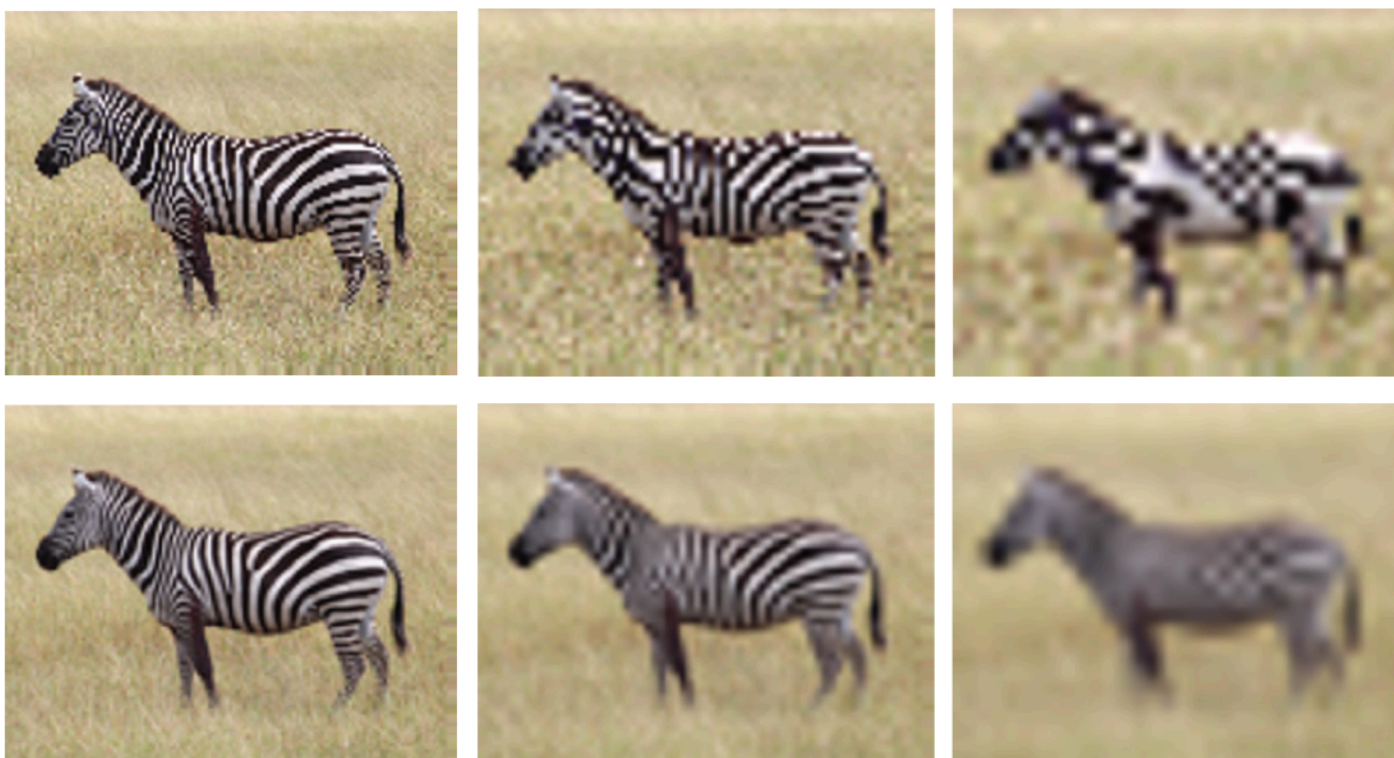
# Antialising filtering

Before sampling, apply a low pass-filter to remove all the frequencies that will produce aliasing.

103×128 | 52×64 | 26×32

Without antialising filter.

With antialising filter.



# Spatio-temporal sampling illusion

# Evidence for filter-based analysis of motion in the human visual system shown via spatio-temporal visual illusion based on sampling

Two potential theories for how humans compute our motion perceptions:

(a) We match the pattern in the image that we see at one moment and compare it with what we see at subsequent times.
(b) We use spatio-temporal filters to measure spatio-temporal energy in order to measure local motion.

This illusion favors one theory over the other.

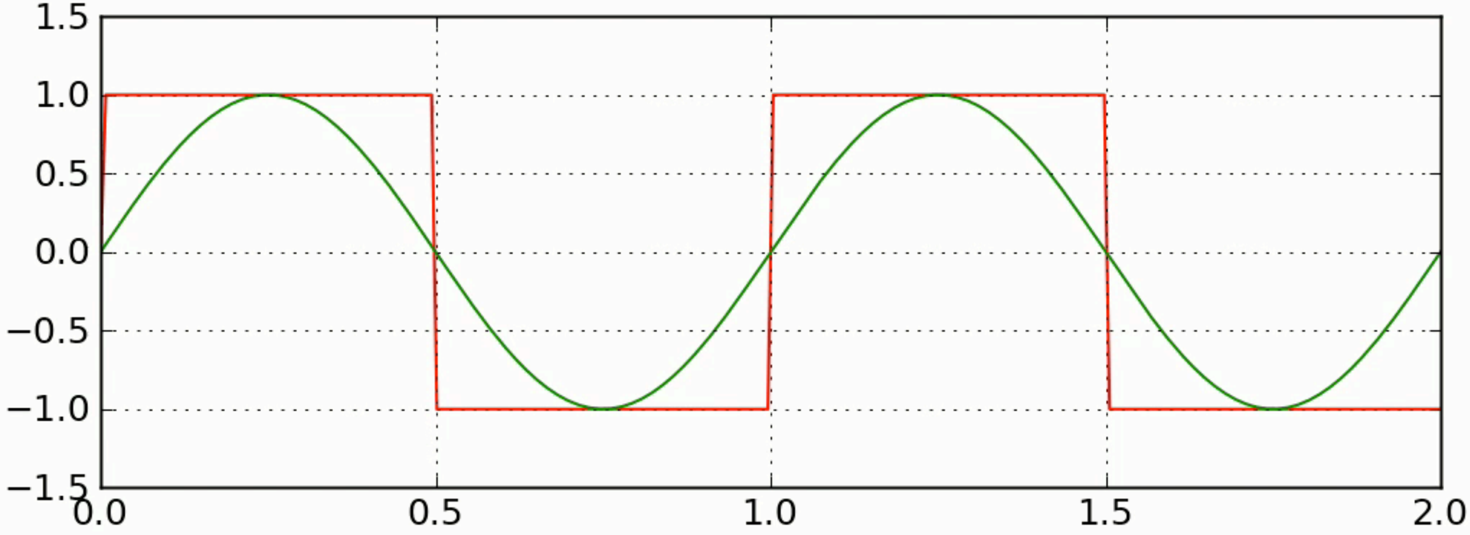
# Square wave Fourier components

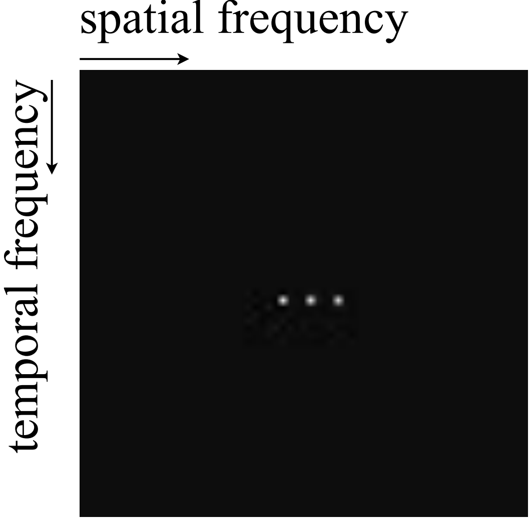
Using Fourier series we can write an ideal square wave as an infinite series of the form

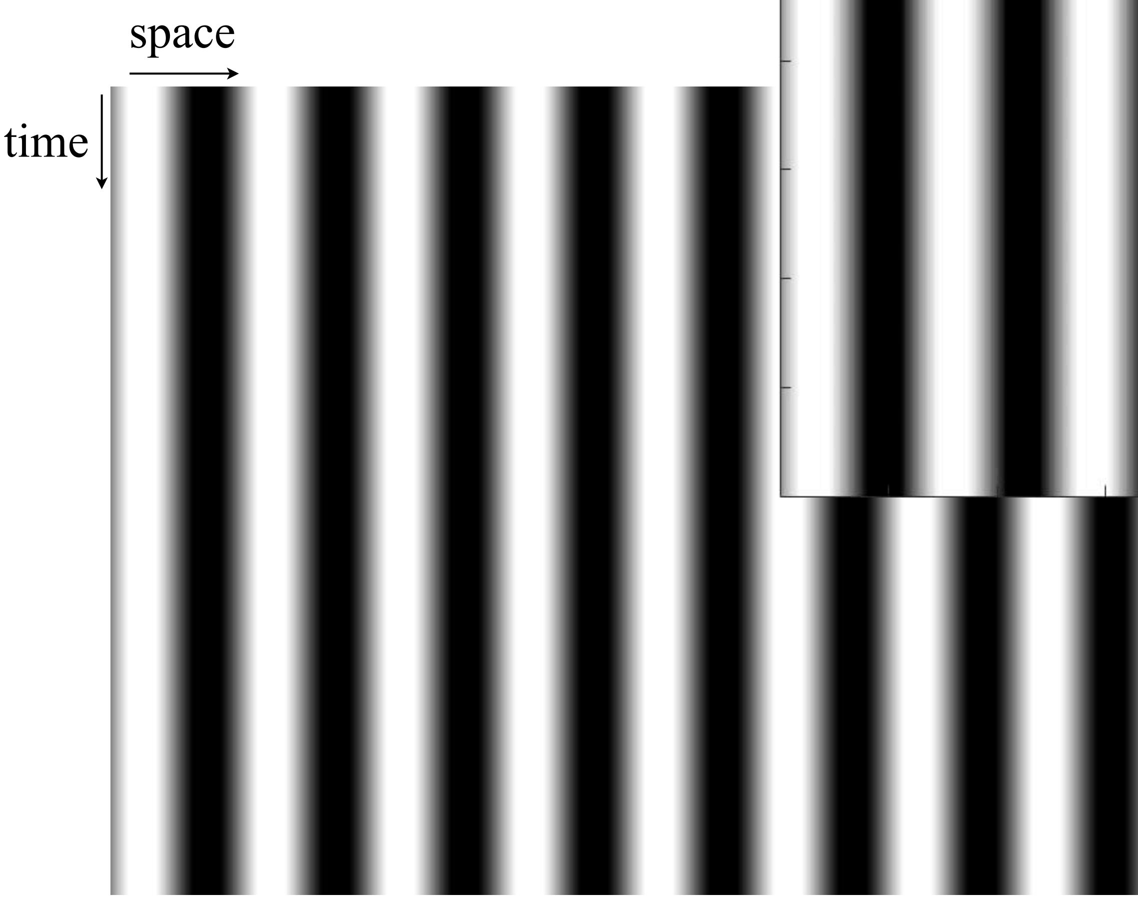
$$x_{\text{square}}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left((2k-1)2\pi f t\right)}{(2k-1)}$$

$$= \frac{4}{\pi}\left(\sin(2\pi f t) + \frac{1}{3}\sin(6\pi f t) + \frac{1}{5}\sin(10\pi f t) + \cdots\right).$$

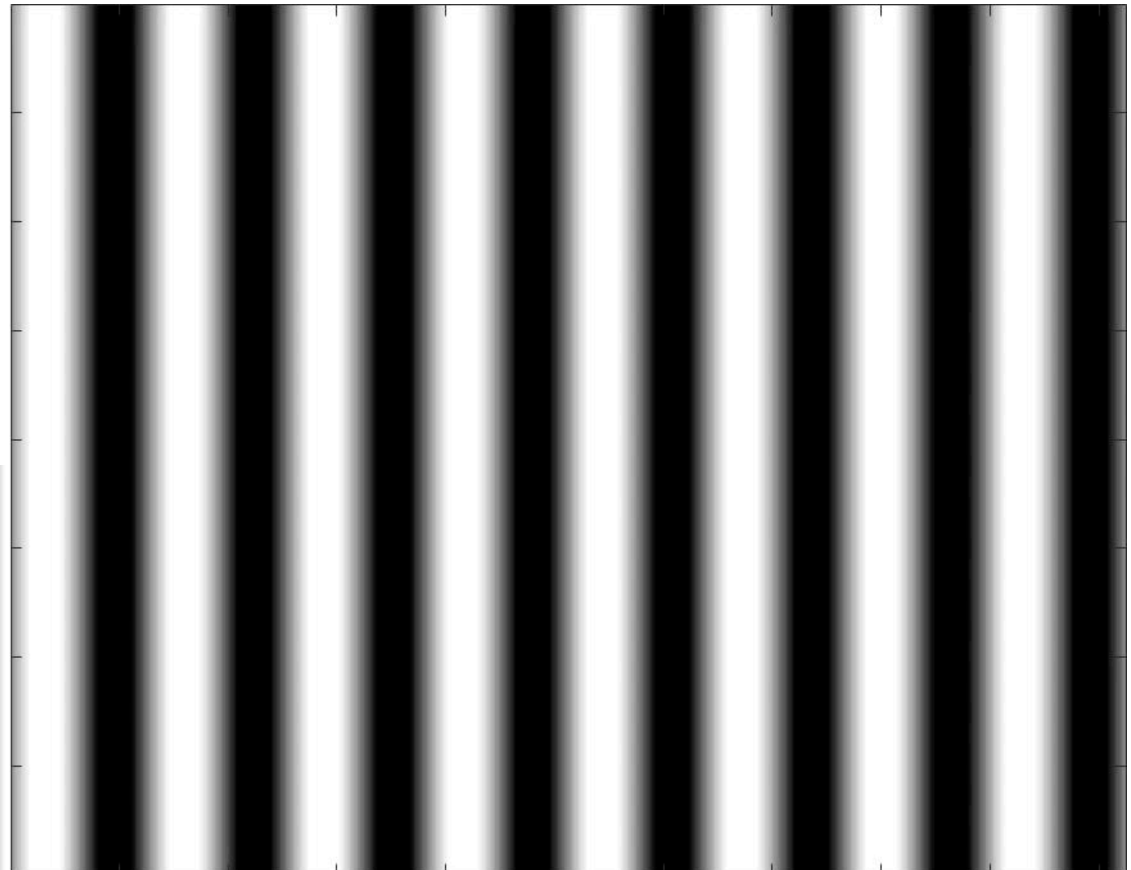
# A square wave is an infinite sum of sinusoids

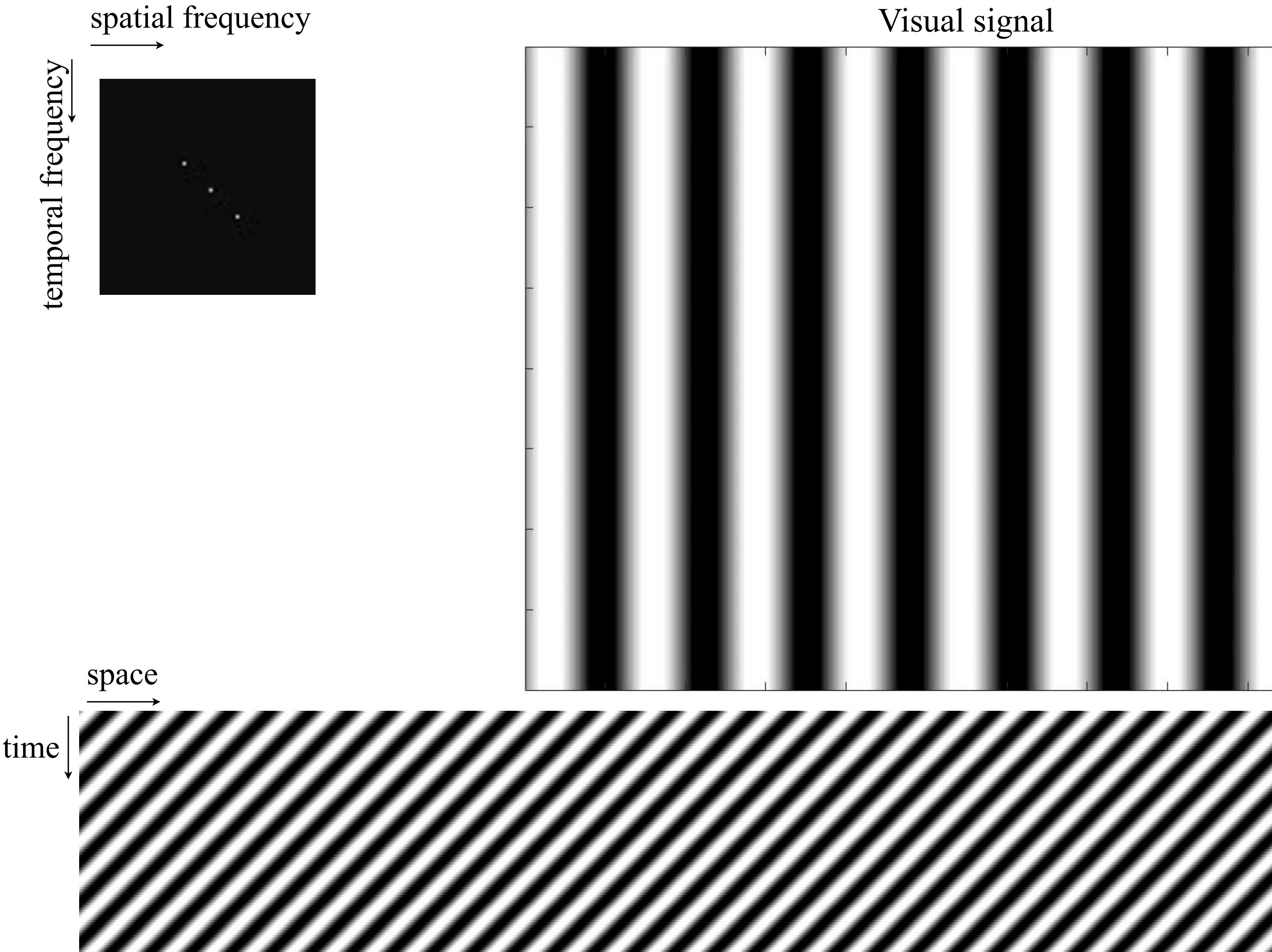


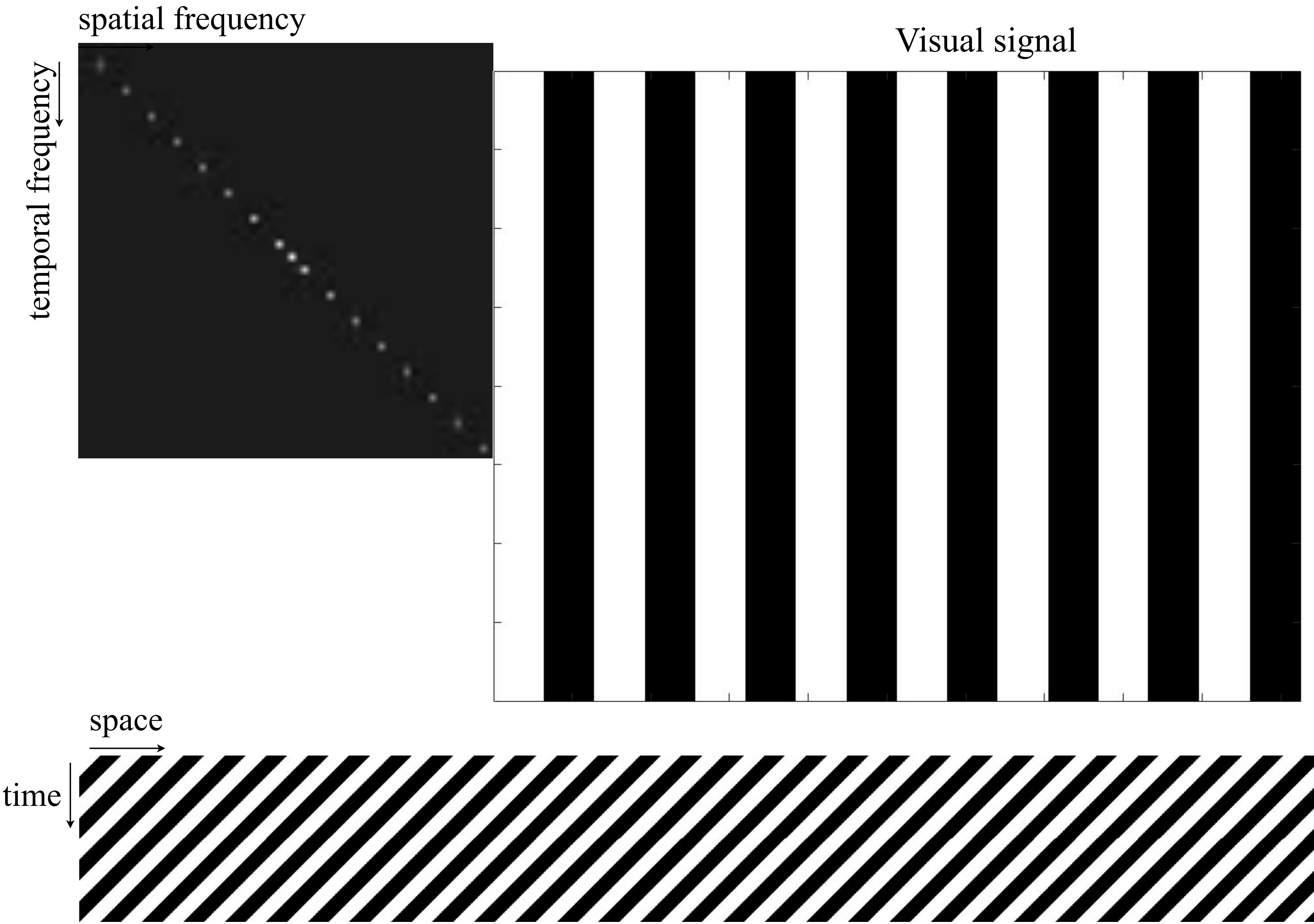


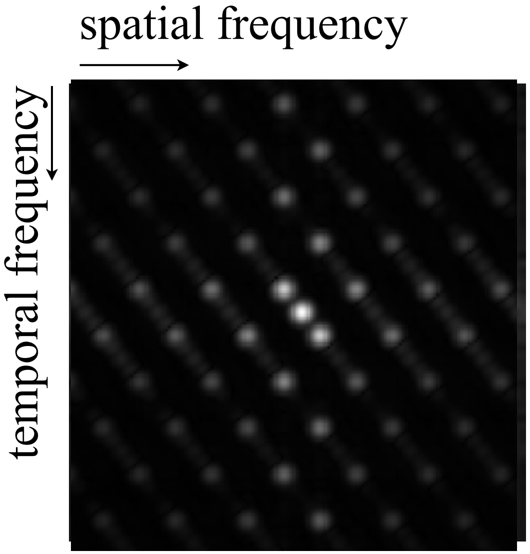

Visual signal

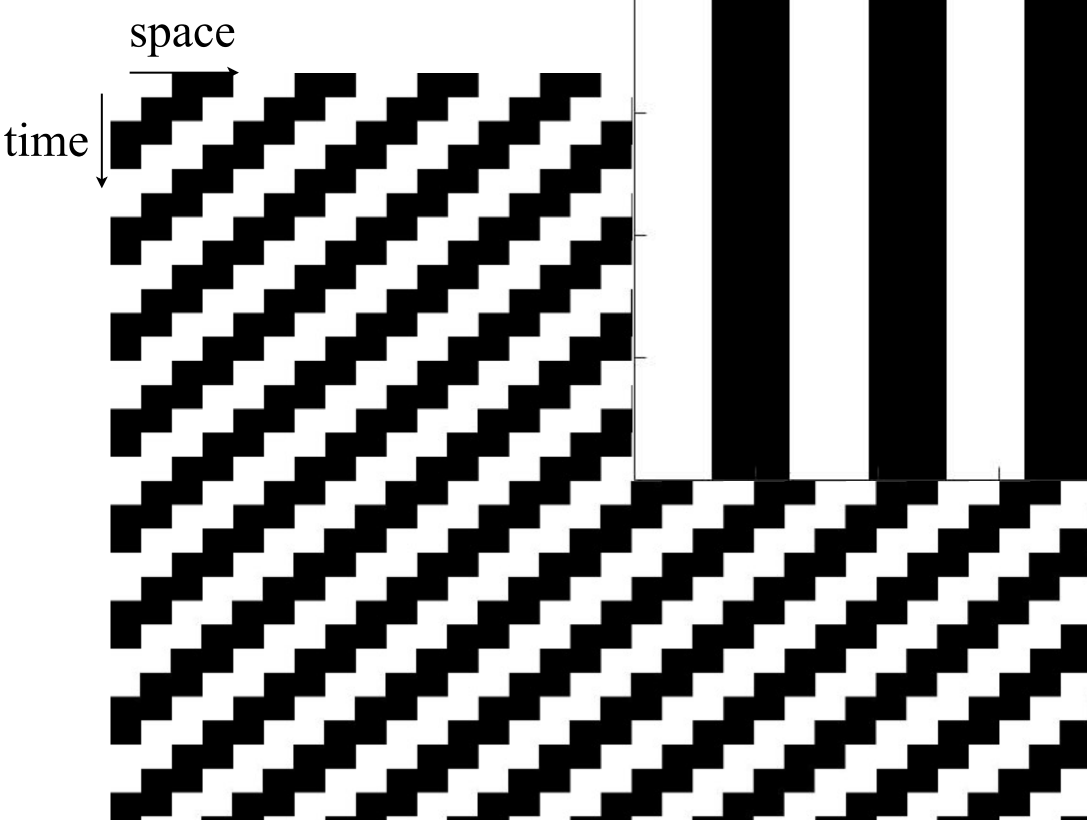


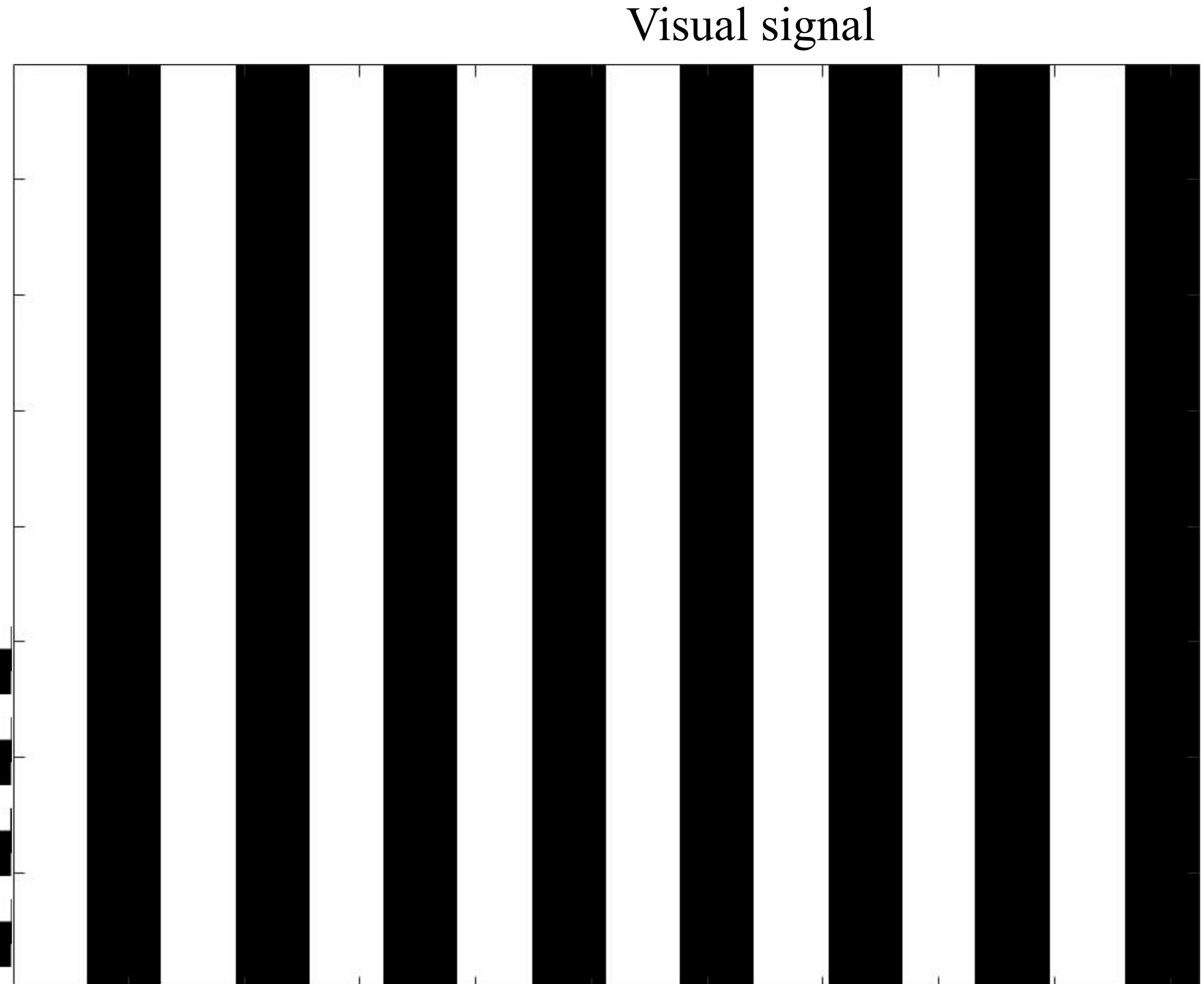

spatial frequency
temporal frequency
Visual signal
space
time

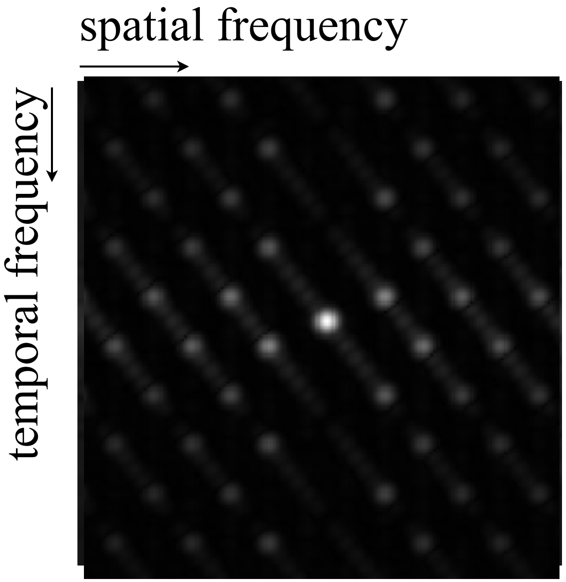


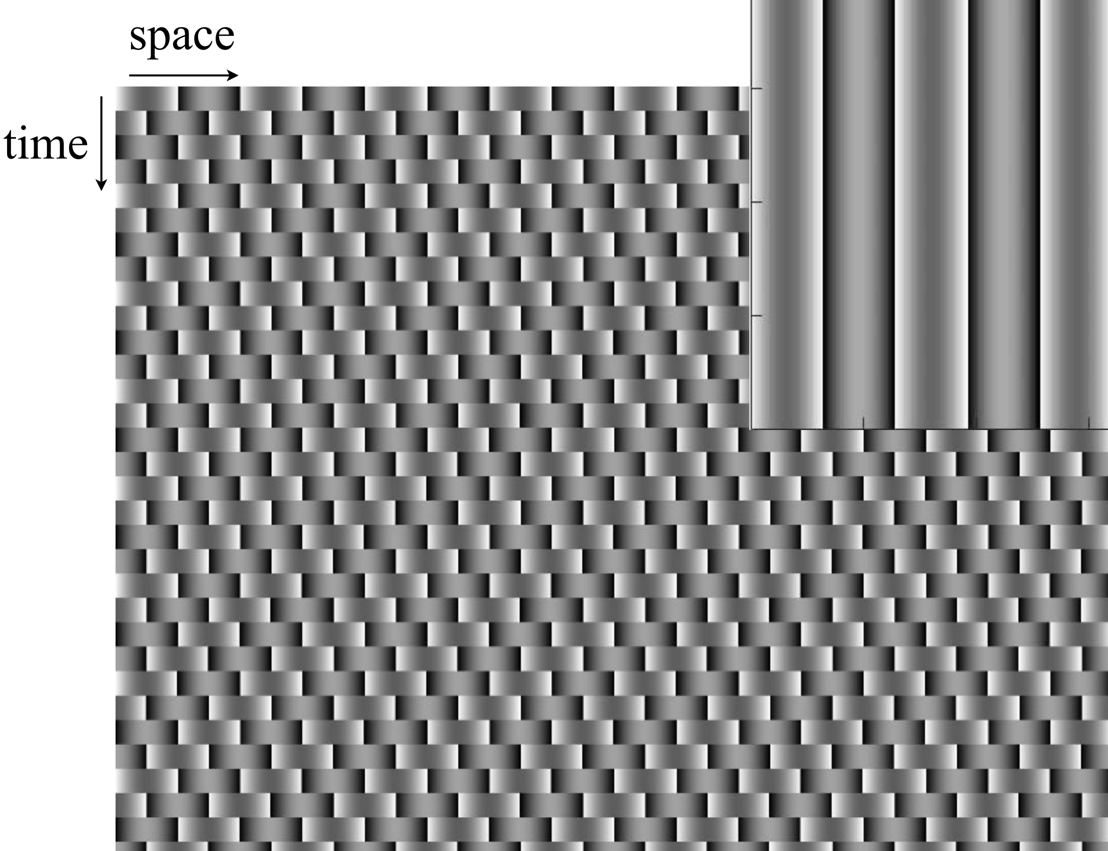

alpha: 1  squareFlag: 1 offset: 4

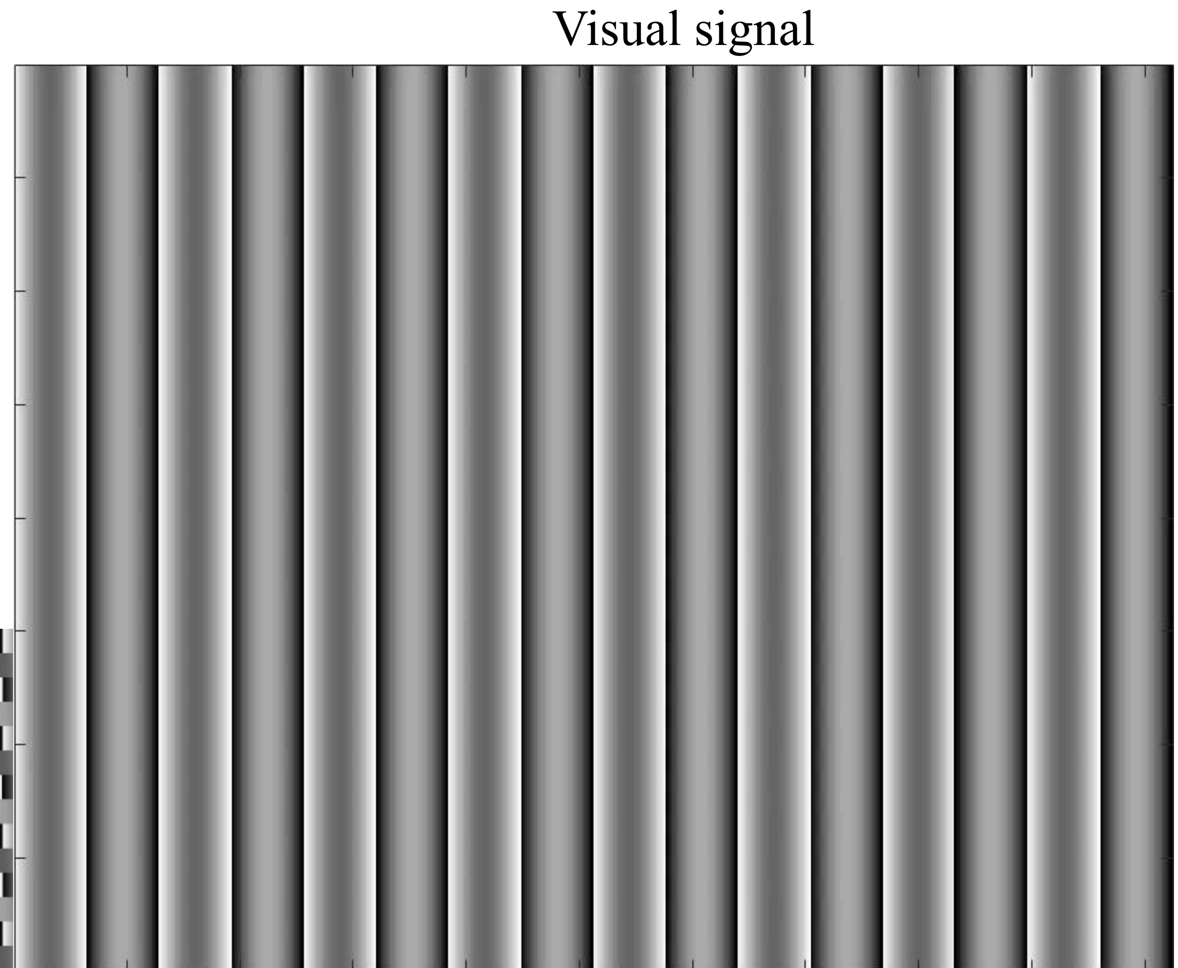

spatial frequency
temporal frequency

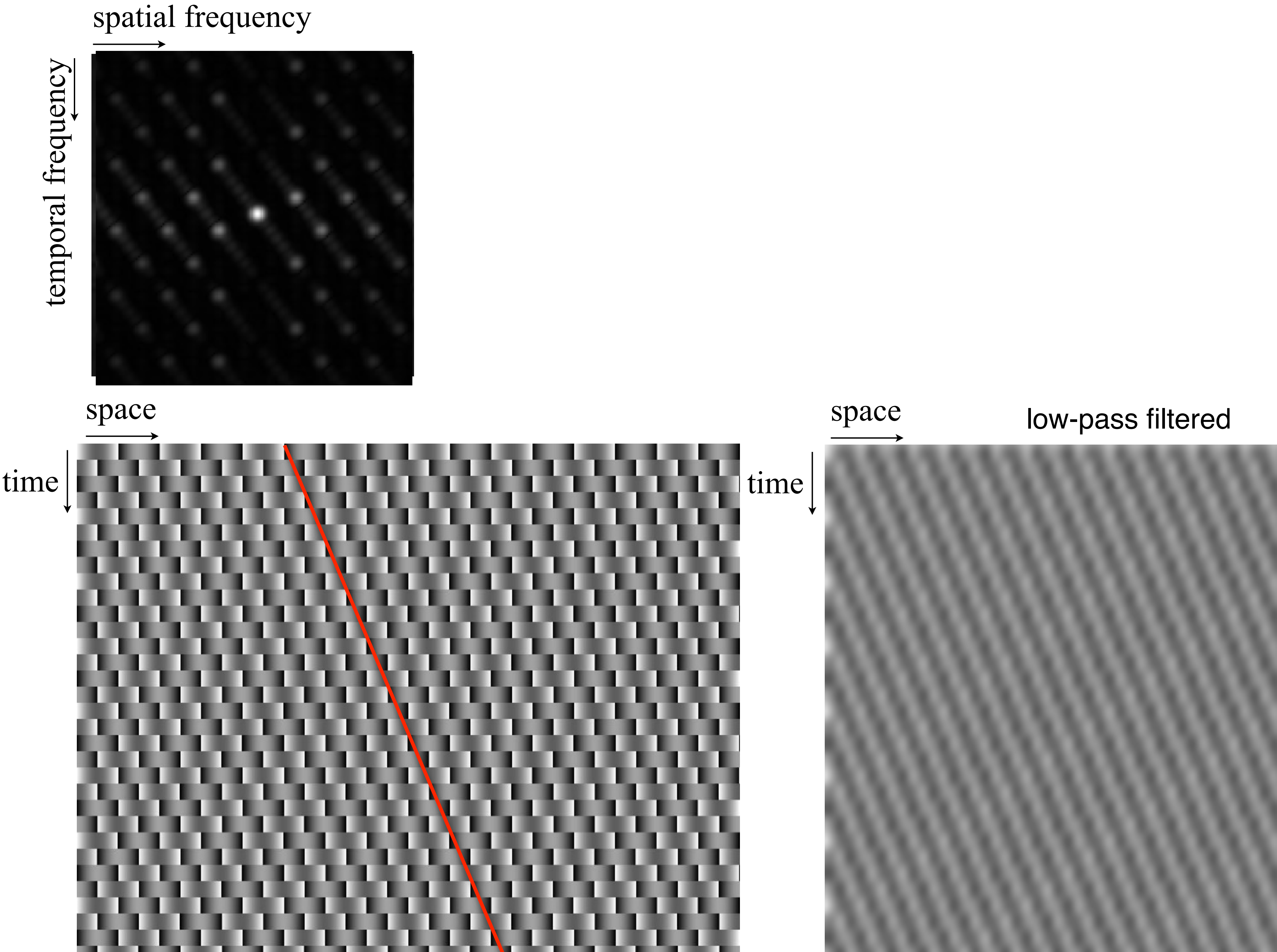

space
time

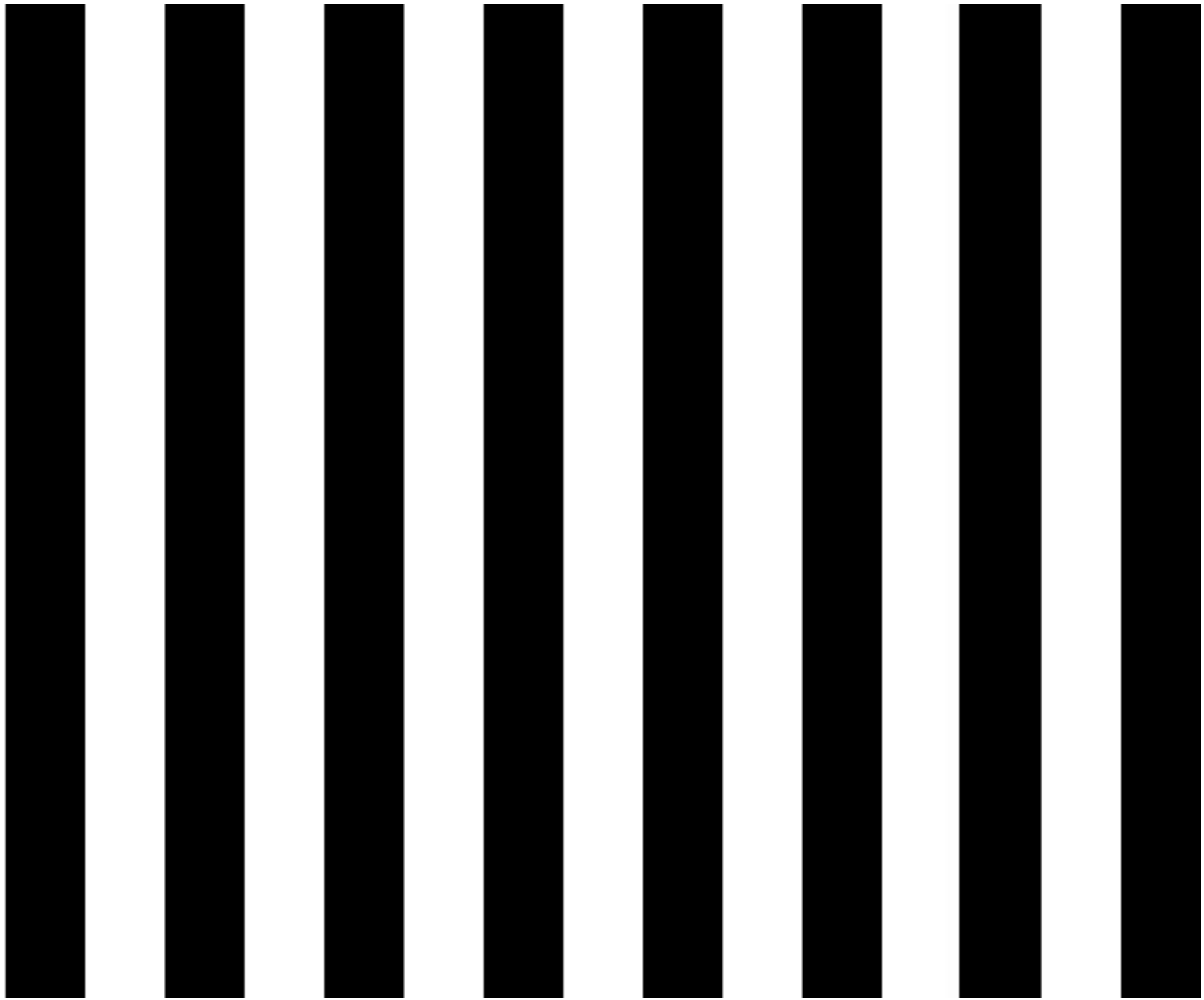

Visual signal

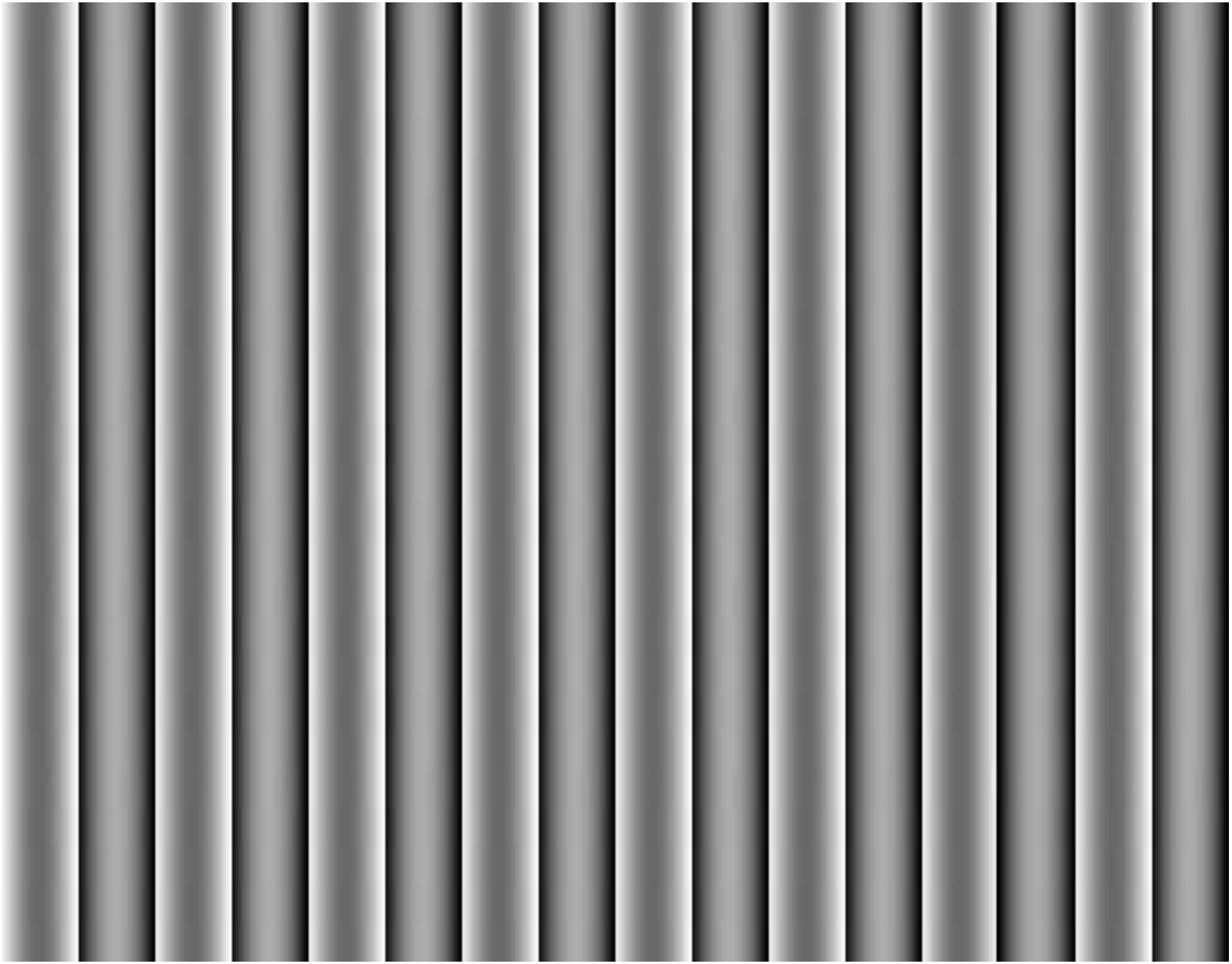

spatial frequency
temporal frequency


space
time

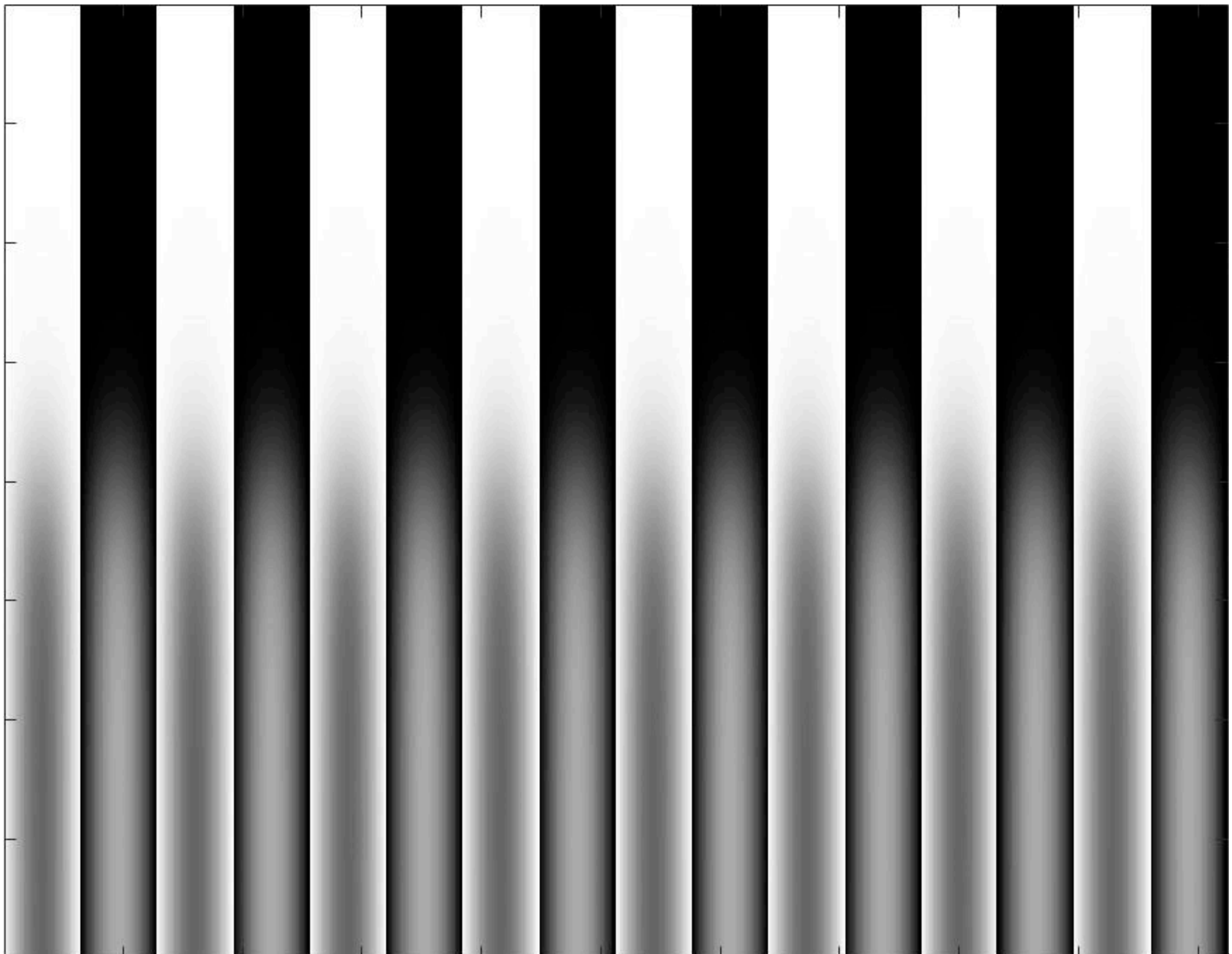

Visual signal


spatial frequency
temporal frequency
space
time
space
low-pass filtered
time

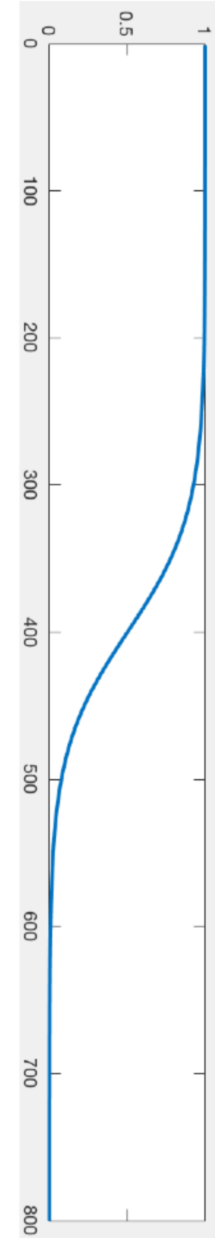




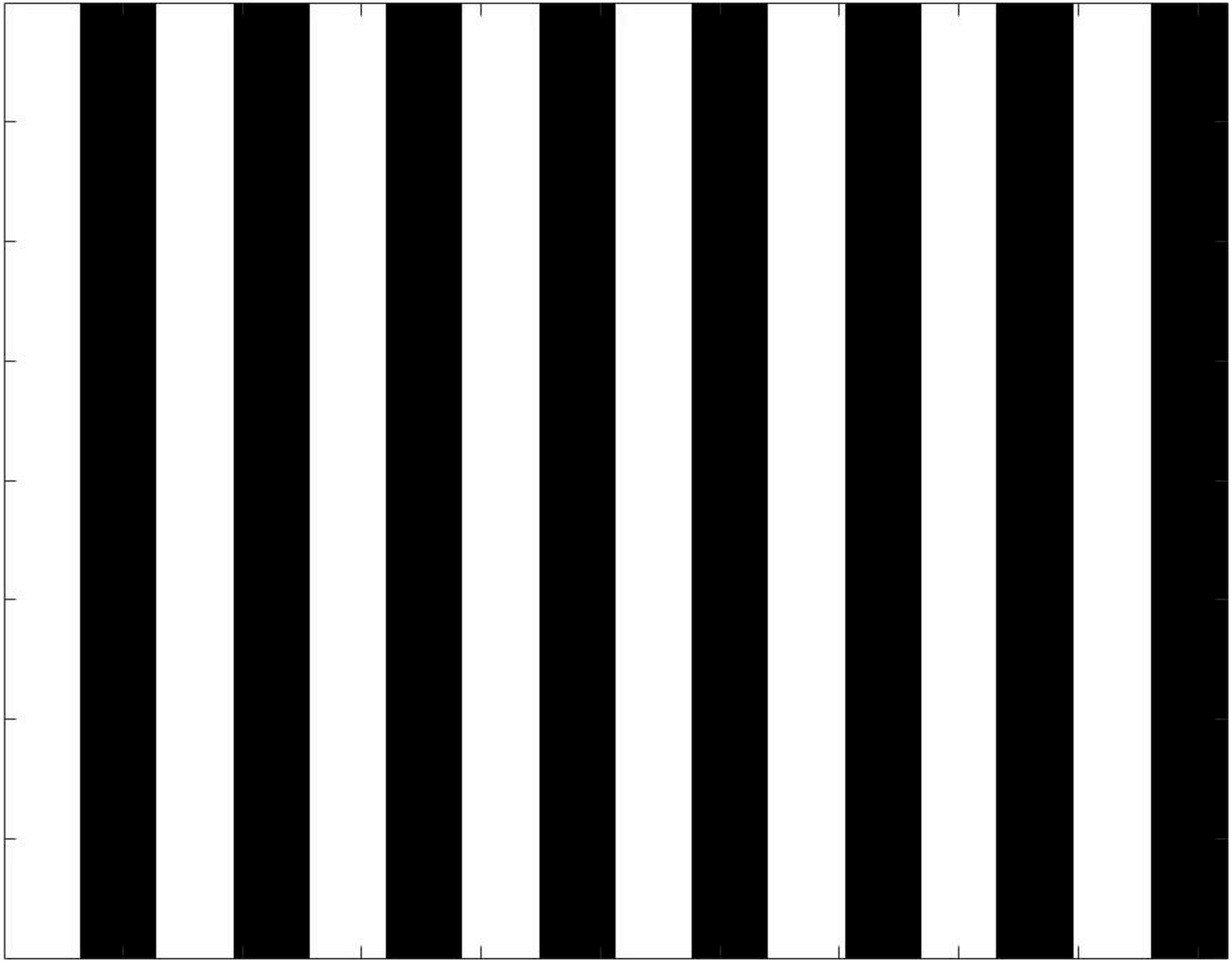
# blend over the two conditions




fraction of square wave
fundamental frequency



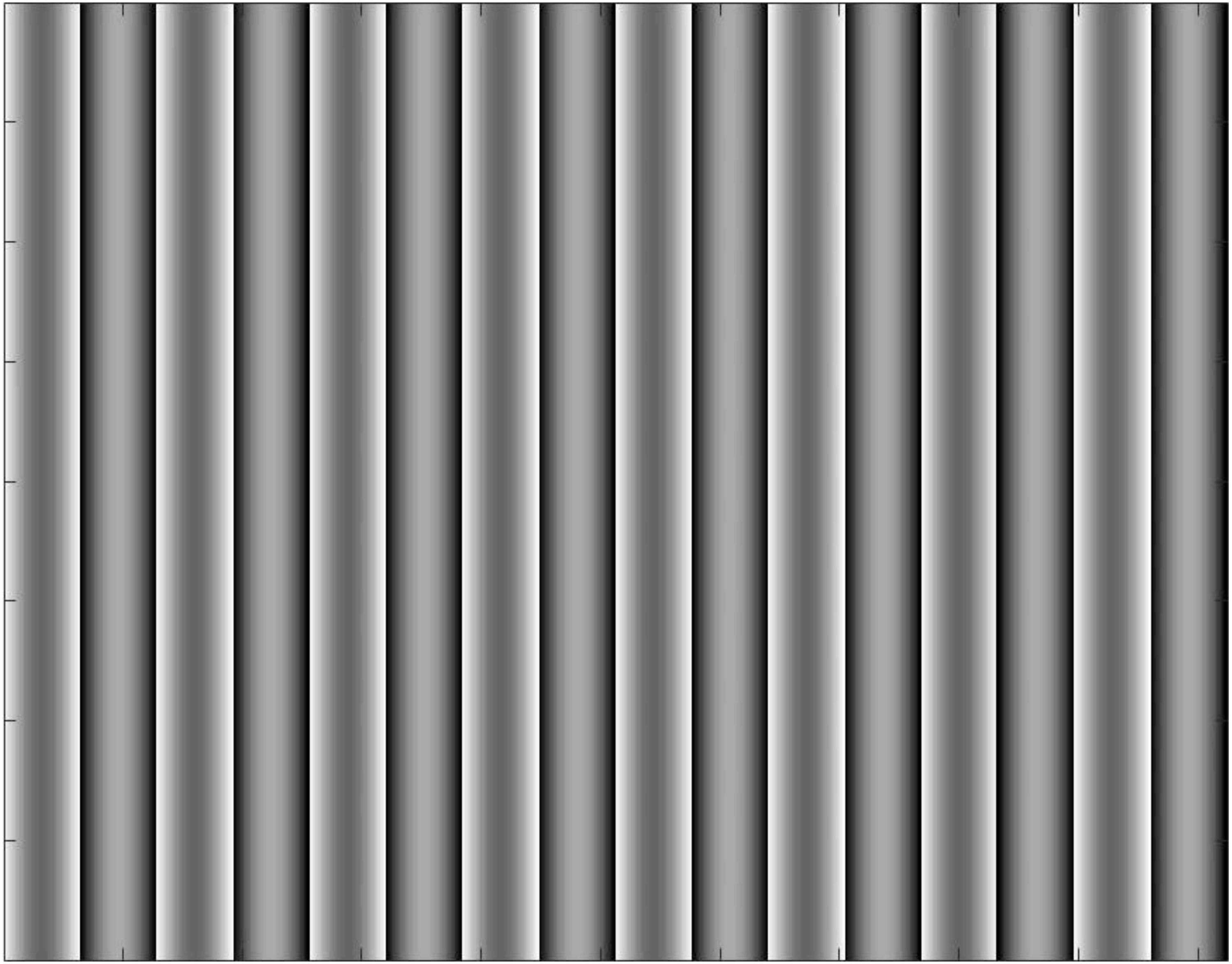
# faster display speed





alpha: 1  squareFlag: 1 offset: period/4

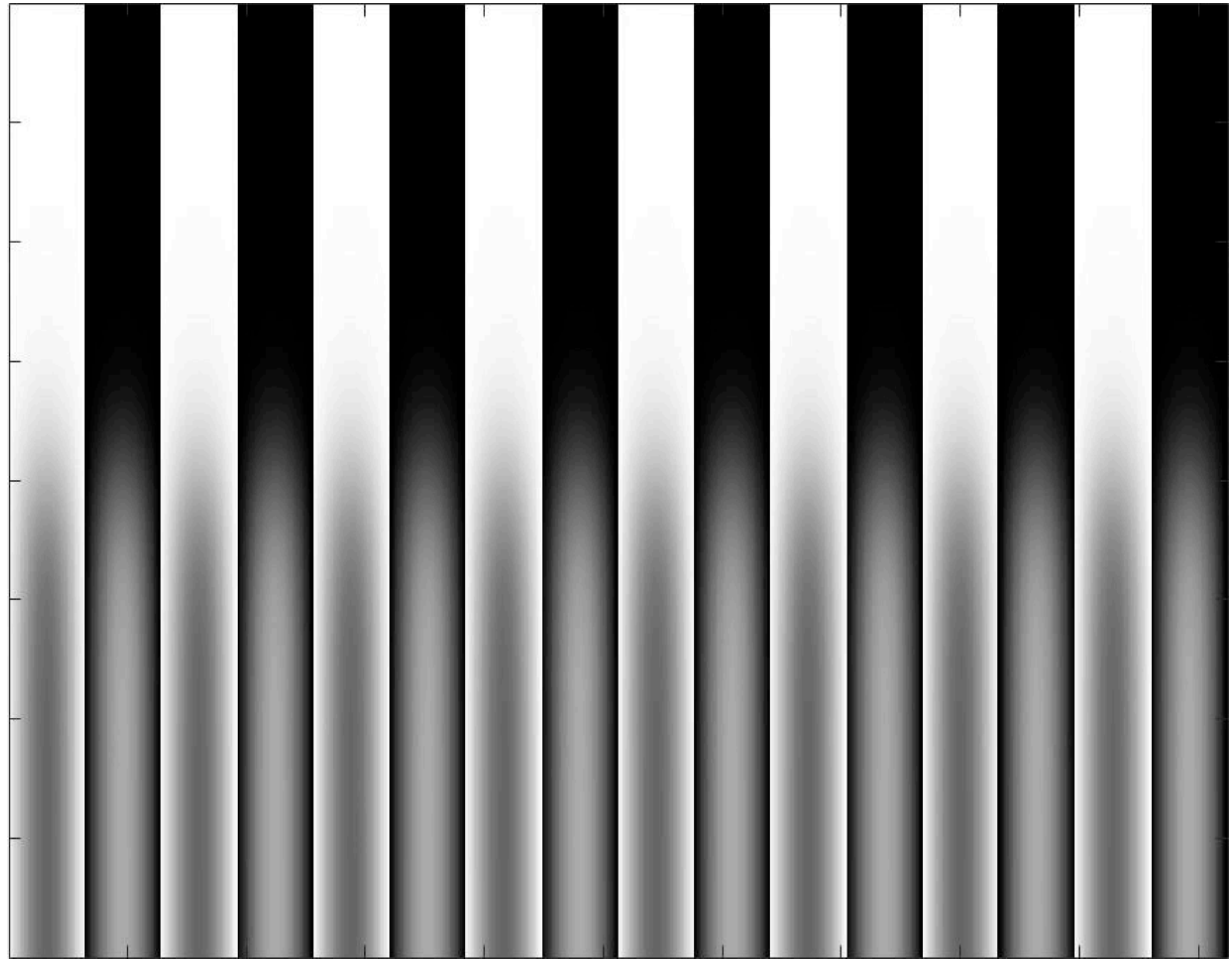
# faster display speed



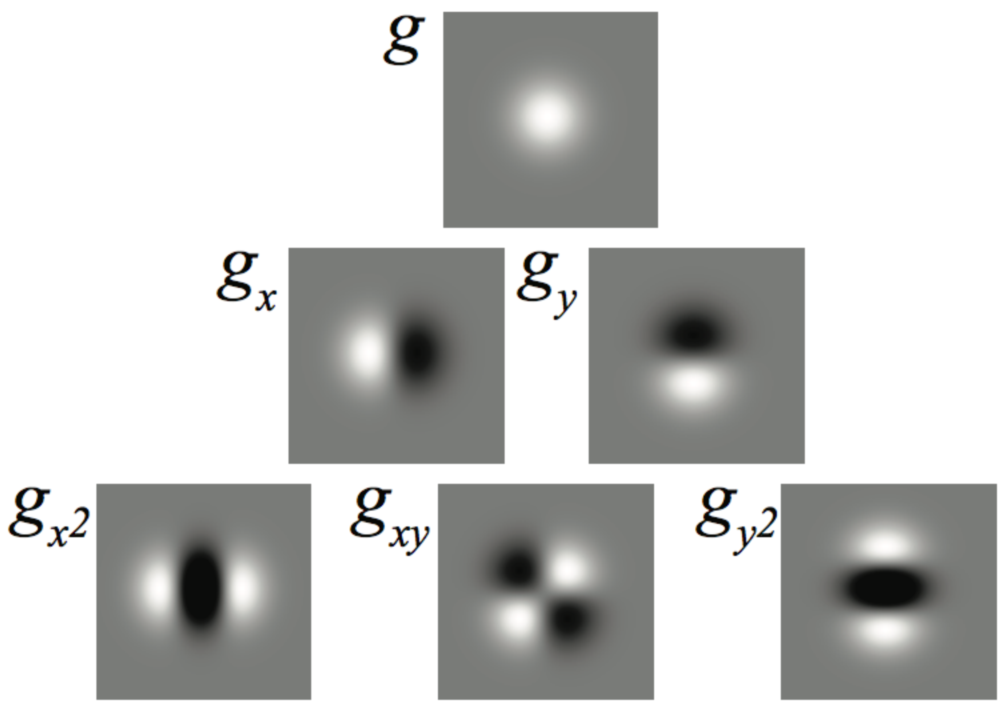


alpha: 0  squareFlag: 1 offset: period/4

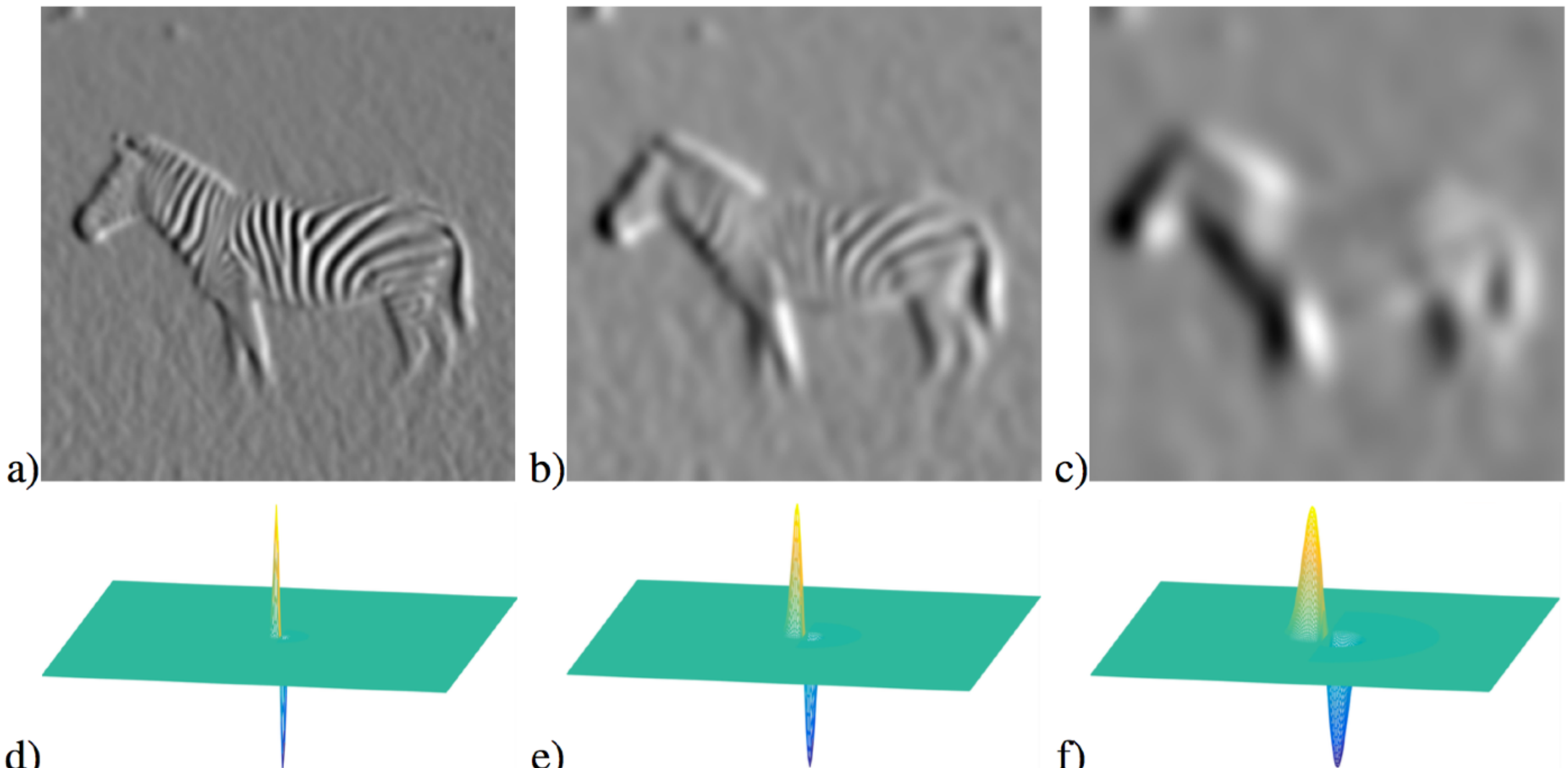
# fast blended…



# derivatives of Gaussians



# derivatives of Gaussians



# Space-time Gaussian derivatives

$$\frac{\partial g}{\partial t} = \frac{-t}{\sigma_t^2} g(x,y,t)$$

$$\nabla g = \left(g_x(x,y,t), g_y(x,y,t), g_t(x,y,t)\right) =$$
$$= \left(-x/\sigma^2, -y/\sigma^2, -t/\sigma_t^2\right) g(x,y,t)$$

**Note**: we can discretize time derivatives in the same way we discretized spatial derivatives. For instance:

$$f[m,n,t] - f[m,n,t-1]$$

# Cancelling moving objects

Can we create a filter that *removes* objects that move at some velocity **(vx , vy)** while keeping the rest?

# Space-time Gaussian derivatives

For a global translation, we can write:

$$f(x, y, t) = f_0(x - v_x t, y - v_y t)$$

Therefore, we can write the temporal derivative of $f$ as a function of the spatial derivatives of $f_0$ :

$$\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} = -v_x \frac{\partial f_0}{\partial x} - v_y \frac{\partial f_0}{\partial y}$$

And from here (using derivatives of $f$, which will be the same):

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} = 0$$

This relation is known as the “Brightness change constraint equation”, introduced by Horn & Schunck in 1981

# Space-time Gaussian derivatives

Can we create a filter that removes objects that move at some velocity **(vx , vy)** while keeping the rest?

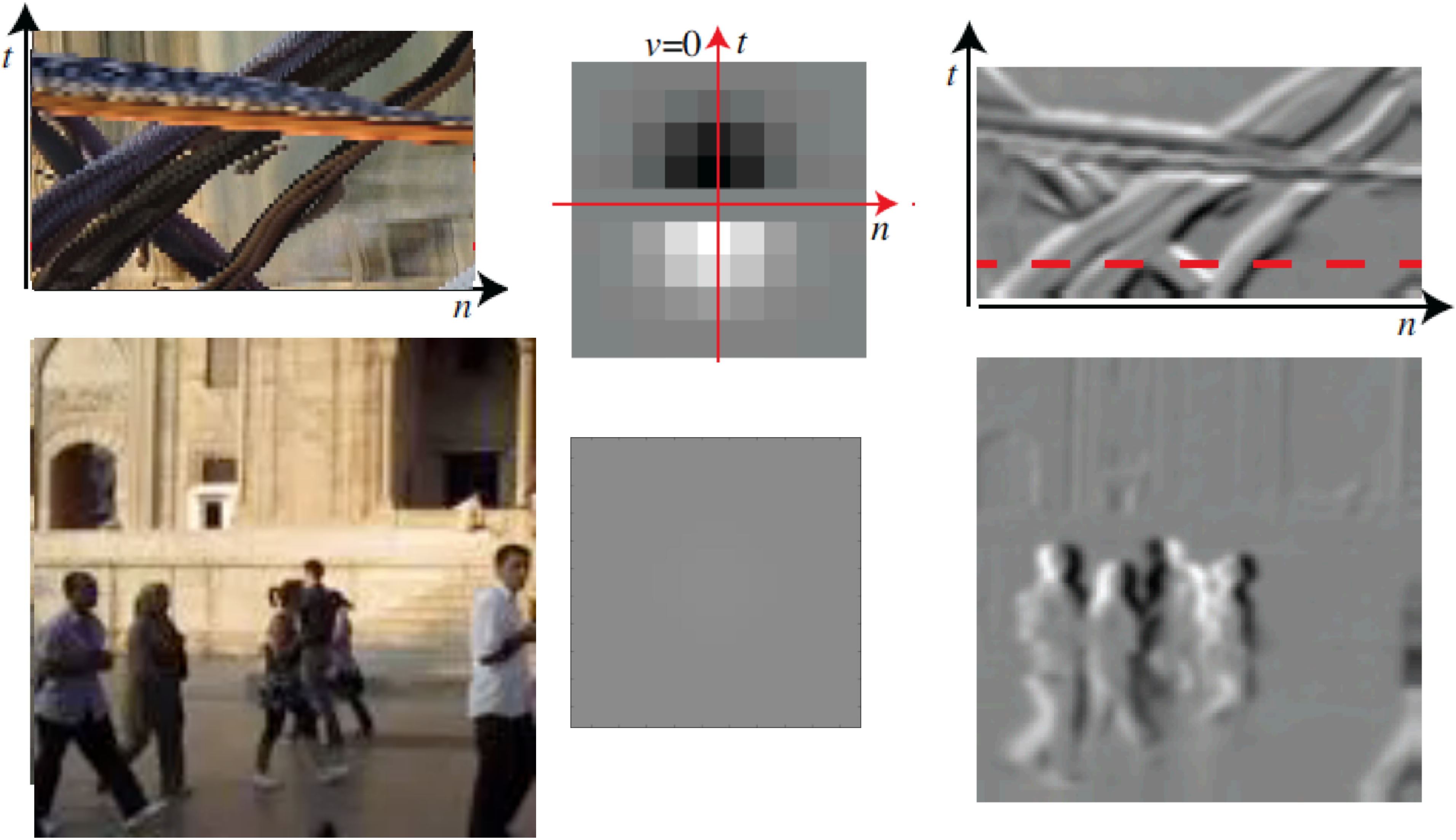
Yes, we could create a filter that implements this constraint:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} = 0$$

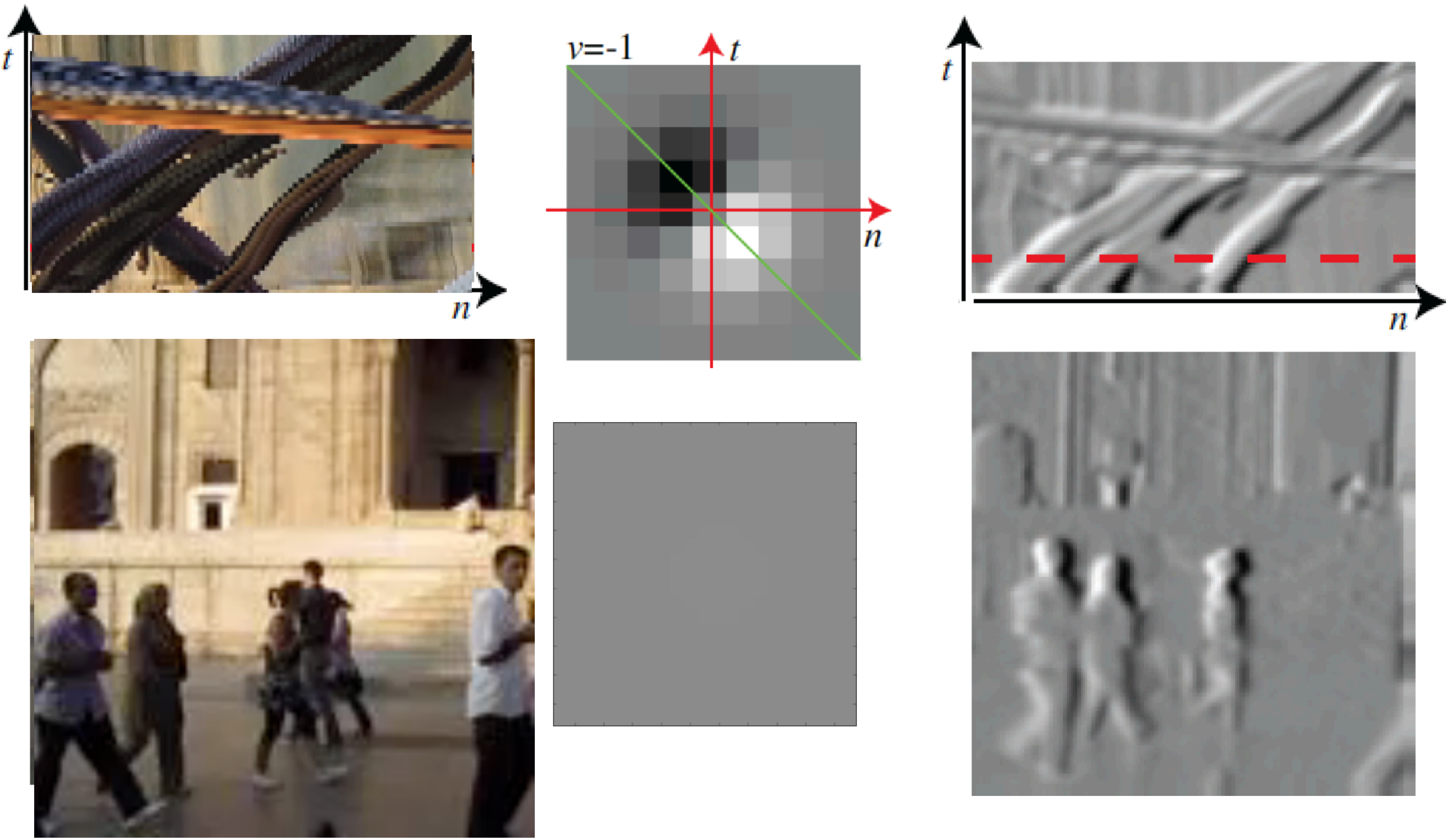
We can create this filter as a combination of Gaussian derivatives:

$$\begin{aligned} h(x, y, t; v_x, v_y) &= g_t + v_x g_x + v_y g_y \\ &= \nabla g \left(1, v_x, v_y\right)^T \end{aligned}$$

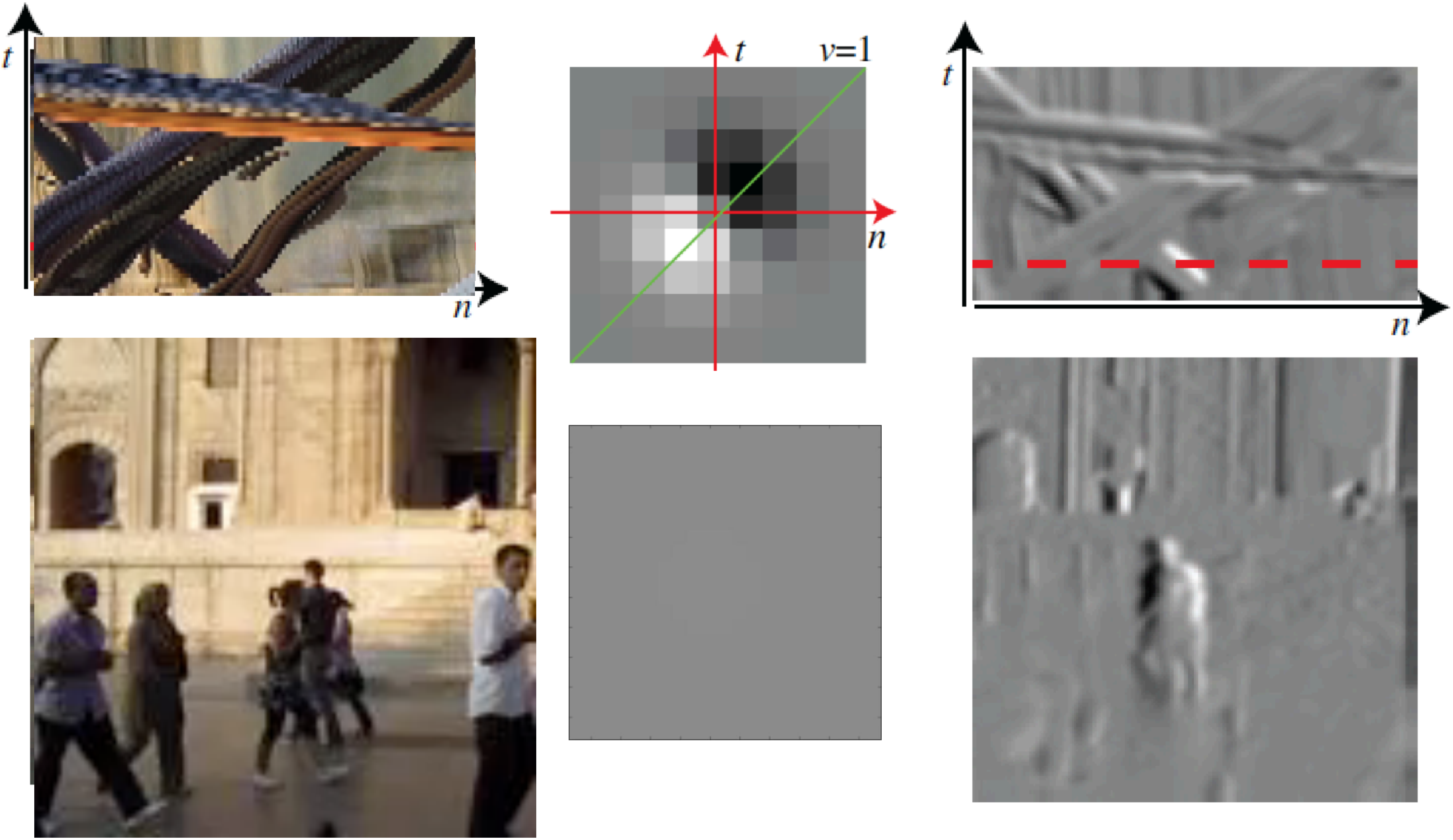
# Space-time Gaussian derivatives




Nulling-out $v_x = 0$, $v_y = 0$ motion




Nulling-out $v_x=-1$, $v_y=0$ motion




Nulling-out $v_x = 1$, $v_y = 0$ motion

# end